THE EFFECTS OF ERRORS IN SURVEYING.

[*Frontispiece.*

CENTRING A THEODOLITE UNDER A ROOF-STATION IN A MINE.

THE EFFECTS OF
ERRORS IN SURVEYING.

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With Frontispiece and 22 Illustrations.

LONDON:
CHARLES GRIFFIN AND COMPANY, LIMITED.
PHILADELPHIA: J. B. LIPPINCOTT COMPANY.
1912.

PREFACE.

Owing to the more all-round training now required of engineers, the time has fortunately gone by when apology had to be made for mathematics in an engineering book. As the treatment throughout this little work is frankly mathematical, the author assumes that his reader, if not a profound mathematician, stands at anyrate on terms of easy familiarity with the more elementary branches of the subject.

The book is chiefly addressed to surveyors whose practical experience is such as will allow them to make fullest use of the methods of analysis which are here developed.

Every-day practice with small instruments is dealt with, rather than geodetic surveying, in order to render the work of greatest service to civil and mining engineers. The diversity of mine-surveying problems renders them particularly suitable for discussion, and excuses the frequent selection of instances from that branch of surveying.

The author will always be glad to hear from any reader who has suggestions to offer, textual mistakes to correct, or who has met with difficulty in following any of the arguments.

Grateful acknowledgment is due to the Council of the Royal Society of Edinburgh, and to the University of Birmingham for permitting the use without reservation of material forming part of a paper read before the Society in February, and part of a thesis submitted to the University in April of last year.

H. B.

Edinburgh, *March*, 1913.

CONTENTS

CHAPTER I.

CHAPTER II.

THE ANALYSIS OF ERROR.

CHAPTER III.

THE BEST SHAPE OF TRIANGLES.

CHAPTER IV.

THE PROPAGATION OF ERROR IN TRAVERSING.

CHAPTER V.

THE APPLICATION OF THE METHODS OF DETERMINING AVERAGE ERROR TO CERTAIN PROBLEMS IN TRAVERSING.

CHAPTER VI.

THE PROPAGATION OF ERROR IN MINOR TRIANGULATION.

CHAPTER VII.

APPENDIX.

NOTATION.

The following uniform notation is employed for the more important quantities dealt with. All errors have the plus-or-minus sign :—

ε or ε_A.	General symbol for average error, whether scalar or vector.
ε_P, ε_S	Probable error and mean-square error respectively.
R.	The sum, or resultant, of two or more vector errors.
a_r, a_s.	The average error in reading a vernier, and the average error of making one sight with a theodolite, respectively; expressed in seconds.
v.	The average error, due to sighting and reading combined, in an angle measured by a theodolite; expressed in radians unless otherwise stated.
u.	The average error in a bearing taken by the magnetic needle; expressed in radians unless otherwise stated.
γ.	The maximum permissible displacement in centre, arising out of the imperfect setting of an instrument over or under a station.
r.	The average displacement in centre.
$T, T_1, T_2 \ldots . T_n$	Traverse angles.
$t, t_1, t_2 \ldots . t$.	Average errors affecting traverse angles.
S.	The angle which is actually measured (in place of T) when the theodolite is displaced in centre.
$\beta_1, \beta_2, \beta_3 \ldots . \beta_n$.	The bearings of traverse lines.
$\beta'_1, \beta'_2, \beta'_3 \ldots . \beta'_n$.	The average errors in the bearings of traverse lines.
$L, L_1, L_2, L_3 \ldots . L_n$.	The lengths of traverse lines, in feet.
$l, l_1, l_2, l_3 \ldots . l_n$.	The average errors in the lengths of traverse lines.

F.	The total length of a traverse.
K.	A coefficient, equal to $l \div \sqrt{L}$.
K_1 and K_2.	Values of K for the chain and steel-band respectively.
A, B, C, ...	Triangulation angles
$a, b, c,$	The lengths of the sides of triangles, being respectively opposite A, B, C, . . ., c being the base-line of the system.
$a_1, b_1, c_1,$	Average errors in the sides $a, b, c,$ . . . . respectively.
$\frac{a_1}{a}, \frac{b_1}{b}, \frac{c_1}{c},$	Average fractional errors in the triangulation lines.
z.	A check-base of a triangulation system.
z_1.	The average error in the calculated length of a check-base.
z_2.	The average error in the measured length of a check-base.
z_3.	The average discrepancy between the calculated and measured length of a check-base.
n.	The number of triangles in a triangulation system, or the number of lines in a traverse.
1, 2, 3, . . . N.	Triangulation stations.
Z_N	The distance of a triangulation station, N, from station 1, the origin of the survey.

CHAPTER I.

INTRODUCTION.

"La théorie des probabilités n'est au fond, que le bon sens réduit au calcul . . . elle donne les aperçus les plus sûrs qui puissent nous guider dans nos jugements."—*Laplace.*

EFFECTS OF ERRORS IN SURVEYING.

CHAPTER I.

INTRODUCTION.

SOMETHING by way of foreword is desirable to a book like this, in which a technical subject is viewed from a very special standpoint. The present short chapter is introduced to explain the purpose of the work, to anticipate some of the difficulties which may arise in the reader's mind as he follows the arguments, and generally to clear for action ; the author also takes the opportunity of directing attention to the practical value of many of the methods developed, and of discussing the nature of the foundations on which those methods stand.

No branch of mathematics is treated in this country less according to its deserts than that known as the Theory of Probabilities, of which the Theory of Errors is a part. The reason for this ostracism by teachers is possibly due, not so much to those who, not having studied probability, consider it a fantastic mental exercise of little worth, as to those others who are adverse from including in a course of pure mathematics a subject so uncompromisingly practical in bearing. To persons engaged in astronomical, physical, or chemical research the theory of errors gives helpful guidance in finding the most probable value of a quantity which has been measured several times, perhaps by different methods and different observers, and in calculating the trustworthiness of such a value. It is in surveying, however, that the theory's utility is most manifest, and where it has the widest scope.

The theory of errors may be said to have been developed very largely by one man, Gauss, who succeeded, by a series of writings issued between 1809 and 1827, in bringing the subject practically into its present form. Basing his work on Gauss' "law of least squares," Bessel evolved some valuable processes for distributing error in triangulation, and was the first to apply them to extensive trigonometrical surveys. Since Bessel's day there has been a good deal of what may be termed intensive writing on the law of least squares and its application to triangulation, and several excellent text-books and papers on the subject have been published on the subject in recent years.* His methods, moreover, have been so widely accepted that it is safe to assert them to have been applied to every important triangulation of the last seventy years.

Notwithstanding the attention paid to the subject of distributing error in triangulation, very little advance has been made in the study of the propagation of error in surveys. This little work forms a contribution—for the most part new—to that study; its purpose is to investigate how errors combine in affecting the accuracy of surveys, in order that rules may be framed to help the surveyor to guard against error, and methods devised to allow him to assess the error likely to occur in any

* To mention three—Merriman's "Elements of the Method of Least Squares" (first edition, 1877), Johnson's "Theory of Errors and Method of Least Squares," Crandall's "Text-book on Geodesy and Least Squares"—all by American writers.

given case in practice. The law of least squares is left totally to one side—firstly, because it has been adequately dealt with by others; and secondly, because it will already form part of the mental machinery of the greater number of those who are interested in this book.

It does not, however, necessarily follow because Bessel's mode of distributing error in triangulation has been studied by most scientific surveyors, and the question of the transmission of error neglected, or because the law of least squares has been expounded by many, and the laws of propagation by few, that the former is of greater value in every-day practice than the latter. "One is lead instinctively," said Le Châtelier, "to ascribe a preponderant importance to the sides of a question that one has studied one's self, or on which one finds the greatest number of printed documents,"—yet in point of fact the importance may be more apparent than real, and the greater bulk of published matter nothing more than evidence as to which is the line of least resistance.

The question of the "spreading" of error is only referred to parenthetically now and then; yet, as the study of propagation must necessarily precede the framing of reasonable rules for distribution of error, it seems not unlikely that the results here deduced may lead to a more ample consideration of that subject in the future, especially in relation to traversing.

Throughout this book the criterion known as the "average error" is made use of instead of the "probable error," which is a more usual test of accuracy in this country. The reasons for this choice are given in Chapter II. Professor Holman also employs the average error in place of either the probable or the mean-square error in his work on the *Precision of Measurements*, while the late Professor Chrystal used it in his classical investigation into the nature of Seiches. The preference for the probable error among British mathematicians and physicists does not seem to be founded on good reason; its very name is unfortunate, and liable to lead to misconception; it is slightly more tedious to determine than mean-square error, and considerably more so than average error. If these three criteria are equally good from the theoretical standpoint, as is actually the case, preference ought to be given, one would think, to that which is easiest to compute.

There are, however, mathematical difficulties to be faced far more real than that of assigning the most suitable mean error to serve as criterion in a discussion of the effects of errors in surveying. Prominent among these stands that due to the fact that errors of all descriptions do not obey the same law. The "exponential law of error" represented by equation (1) of Chapter II., and exemplified in the error of sighting a theodolite, while no doubt the chief, is not the only law. The errors arising from reading a vernier, as Holman has pointed out, cannot be referred to the exponential law, nor, it may be added, can those due to imperfect centring of an instrument; yet both these sources are of importance to the surveyor.* Our purpose here is to combine the

* The errors arising from reading a vernier and from eccentric centring have one very marked difference from those obeying the exponential law—namely, that each admits of, and indeed is assessed from, a definite *maximum* error. The continuous function $y = ke^{-h^2x^2}$ precludes, of course, all consideration of such a maximum. The vernier gives rise to a species of error in which the average error is the same as the probable error, being equal to one-half the maximum. In the case of eccentric centring the average is two-thirds of the maximum error, and errors approaching the maximum are more probable than those approaching zero. We, therefore, see that, while these two latter species differ from each other, they diverge still more strikingly from the normal type ruled by the exponential law.

As it seems not unlikely that other species may be found in other branches of applied science, the question would appear not unworthy of mathematical research.

effects of these three species of error, and, to be able to do so, one is obliged to employ methods which are only strictly applicable to the first species. A second difficulty arises from the occasional combination of real and apparent error, as, for example, the real error in angular measurement and the apparent error in base measurement in a triangulation survey.

Still a third difficulty may be mentioned—namely, that of determining the combined effect of two or more independent errors, acting in different directions, and simultaneously disturbing the position of a point. This problem is solved by the theorem on vector errors (§§ 16, 17, 18), which the author believes was first enunciated in a paper he read before the Royal Society of Edinburgh early in 1911.* He is inclined to think that that simple method possesses a scope by no means bounded by the limits of this little book, and that it is likely to yield results of some value to the astronomer as well as to the surveyor.

In dealing with probability generally, and particularly when difficulties of a peculiar nature enter, it is always more troublesome to lay down trustworthy premises than to build mathematical structures upon them, and it would seem that the most satisfactory method of testing whether the premises are sound is to compare one's results as often as possible with those derived from experience. This test is applied at several points in the subsequent chapters, and indeed not a few of the results obtained turn out to be nothing else than algebraic statements of facts which have long been known in a general and more or less shadowy way to surveyors. In the last chapter, where the results are massed together, their practical bearing is emphasised, partly to show to what extent they conform with empirical conclusions.

We are therefore permitted to say—notwithstanding

* *Trans. Roy. Soc. Edin.*, vol. xlvii., part iv., p. 849

the fine field afforded for criticism of the rigidly theoretical type—that the results attained in this essay are sufficiently near the truth for practical purposes. They are advanced, not because they are mathematically precise, but in the hope that they will serve as *guides* in surveying practice.

Mention has still to be made of the methods of analysing error in actual surveys, as exemplified in Chapter V. These methods permit of answers being reached to some of the most difficult questions in surveying. When a surveyor is about to commence an important piece of work, and has settled on the degree of accuracy he desires to attain, the following questions call for solution :—(1) What field methods give a reasonable likelihood of reaching this degree of precision? (2) If such-and-such methods are used, what are the odds that the desired precision will be reached or exceeded? (3) In what way is the result to be achieved with a minimum expenditure of labour and time?

In present-day practice, answers to these questions can only be supplied from experience, which is always difficult to correlate particularly in traversing. Straightway to answer the second question correctly would need, indeed, such a weight of carefully digested experience that few will be found willing to attempt it.

The methods given in the following pages, leading up from data gathered from experience, supply definite answers to the first two questions, and give considerable help towards solving the third. By means of them it is possible to calculate before commencing a survey—not the *actual* error which will result, for that is impossible—but the error which may be expected *on the average*. This can be done for triangulation, and for traverses run by theodolite, miner's dial, or other instrument. After the average error has been obtained, a table given in the Appendix allows, further, of the chances being roughly weighed of any stated degree of precision being reached in the survey by the field methods under consideration. A determination of this kind is especially to the point when, as in the mines of the Transvaal, surveys are required by law to be correct within stated limits of error.*

To supply a conclusive answer to the third of the above queries would need a knowledge of the ability of the particular engineer, and of the conditions under which he works: it is, therefore, more of an individual matter than either of the other two. Yet attention is often drawn, in computing the average error in a survey, to the fact that the error is being produced more, perhaps, from inaccuracies in linear than in angular measurement, and in such circumstances it is permissible to infer that some less exact and, therefore, more rapid method

* Clause 110, Section X., of an Amendment to Regulations issued under the "Mines, Works, and Machinery Ordinance, 1903," of the Transvaal, reads:—

> Errors in mine measurements shall be judged according to the importance of the measurements in question. The following shall be considered as rules in making allowances for errors:—
>
> (*a*) The length of the line joining the positions of any given point, as determined by the beginning and closing of a traverse, shall not exceed $\frac{1}{1000}$ of the sum of the lines used in such traverse, and the error in the measurement of a line shall not exceed $\frac{1}{1000}$ of its true length, nor shall the error of the measurement of its direction in reference to the axes of co-ordinates exceed four minutes of arc.
>
> (*b*) In taking levels in the mine the maximum of error shall not exceed $\frac{1}{5000}$ of the horizontal length.
>
> (*c*) In special measurements that have for their object the fixing of the positions of shafts to be sunk, and the establishment of connections, the admissible error shall not exceed the half of the allowance for errors given above.
>
> The Government Mining Engineer may, in any case where he deems it necessary, cause a check survey to be made. The cost of such survey shall be borne by the owner of the mine, where it is proved that the error of the survey by which the plan was constructed exceeds the above-mentioned limits of error.

could be used in measuring angles without appreciably increasing the resulting error. Such a discovery is always of value, if made use of, owing to the time saved. It may be remarked here how disproportionate are the time and accuracy factors. To double the precision of measurement with a given instrument will generally require quite four times the expenditure of time, and sometimes considerably more.

In comparing two rival field methods from the point of view of expense, it ought not to be forgotten that one of the chief items of cost is that of assistance ; so that the method likely to turn out the cheaper is that which saves time in the field though at the expense of additional time in the office. Expeditious field methods are especially to be sought after in mine surveying, since the increasing pressure of work throughout the twenty-four hours makes the surveyor more and more of a nuisance underground.

No process of calculation, then, which enables one to find out how time may be saved in the field without appreciable reduction of accuracy, can be considered as being without value.

The tedium of computing the average error affecting lengthy surveys is relieved to a large extent by making use of the tables of the Appendix.

CHAPTER II.

THE ANALYSIS OF ERROR.

"Les observations et les expériences les plus précises sont toujours sujettes à des erreurs qui influent sur la valeur des élémens que l'on veut en déduire."—*Laplace.*

"The more the surveyor knows about the sources and nature of errors the more likely he is to judge correctly of their relative importance."—*Tracy.*

CHAPTER II.

THE ANALYSIS OF ERROR.

1. **Probability.**—It is equally probable that a coin tossed fairly into the air will fall "heads" as "tails." It is certain (setting aside the chance of it balancing on edge), that the result will be either "heads" *or* "tails"; therefore, if we represent certainty by the numeral 1, we may state the probability of "heads" turning up in any single throw as $\frac{1}{2}$.

Another common way of representing the chance in question is to say that the "odds are even" that the coin will fall "heads."

The odds are 3 to 1 against hearts being shown at a single cut of a well-shuffled pack of cards, or the probability is $\frac{1}{4}$. Similarly, the odds against the card shown being the King of Hearts is 51 to 1, and the probability of the event $\frac{1}{52}$.

We see that when there is any uncertainty whatever about an event, the probability of its occurrence must always be a fraction less than unity; and, further, that that fraction is made up of a denominator representing the total number of possible chances (all supposed to be equally possible), and a numerator representing the likelihood of the occurrence amongst all those possible chances. The odds *against* an event are measured by the ratio of the degree of likelihood that it will not occur to the degree of likelihood that it will occur; while the odds *in favour* of an event are the reverse of those against it.

2. **Error.**—Any quantity determined by observation, such as an angle or length in surveying, can never be found with absolute accuracy. Error is always present in results so derived. We may here distinguish between three classes of error—namely, mistakes, cumulative errors, and accidental errors.

Mistakes, or **gross errors,** as they are occasionally called, arise through carelessness or inattention. They are more frequent with unskilled than with skilled observers, and occur more often when the observer is tired. In this class may be placed such avoidable errors as the slipping of a parallel rule in protractor plotting, confusing the 40 mark with the 60 mark of a chain, or entering the length of a line as 12 chains when actually it is 13. In surveying, mistakes of this kind are guarded against by providing checks on the work, and, as a rule, the greater their magnitude the more easily are they discovered.

Cumulative, or systematic, errors are those which always influence the result in one direction, making it either always too large or always too small. For example, the error which arises from disregarding slope in chaining is of this nature; it is a positive error, since it always makes the result too large. To use a chain which is, say, one link too long is to give rise to a negative error; at the end of the first chain-length the error will be one link, at the end of the fourth chain-length four links; thus the error is evidently of a cumulative character. The errors resulting from using a tacheometer having wrongly spaced webs, or from using vertical angles influenced by index error are other instances of this class. All such errors, however, due, as they are, to an instrument being out of adjustment, or to the neglect of obvious precautions, can be avoided or can be corrected for, and so they need not concern us further here.

Accidental errors result from personal and instrumental imperfections; they can never be entirely eliminated, and cannot be calculated *a priori*, though properly analysed experience permits us to assess them more or less closely.

If we were to measure the three angles of a plane triangle many times with a well-adjusted theodolite, and to add the three angles together after each determination, we should always find that the total differed from 180° by a small amount, no matter how carefully we did the work. The first total may, for example, be 180° 00′ 20″, the second one 179° 59′ 40″, and the third 179° 59′ 50″, when the errors of summation would respectively be + 20″, − 20″, and − 10″. A great number of such determinations would convince us of the truth of the following axioms :—

1. Very large accidental errors do not occur ;
2. Small errors are more frequent than large ones ;
3. Positive errors are as frequent as negative ones.

It is the purpose of this book to discuss the effects of accidental errors in surveys, and in future when the word error is used it must be taken as meaning accidental error, unless it is stated otherwise.

3. **The Probability of Error.**—If small errors are more frequent than large errors, as has just been said, small errors will be more *probable* than large ones.

When the errors are obtained of a very large number of equally trustworthy observations of the same quantity, and their magnitudes plotted as abscissæ against their frequency of occurrence as ordinates, a curve similar to that of Fig. 1 results. It is never so regular as our figure, firstly, because it is very seldom possible to eliminate entirely the effects of cumulative error, and, secondly, because one cannot obtain an infinite number of measurements of the quantity in question. The agreement between a theoretically perfect curve, such as Fig. 1, and that constructed from actual observations, however, has often proved remarkably close.

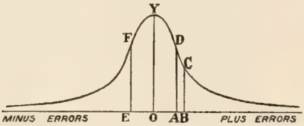

Fig. 1.

This curve (Fig. 1), which is known as the *Probability Curve*, or *Curve of Errors*, is the graphical representation of the relation :—

$$y = ke^{-h^2x^2}, \quad . \quad . \quad . \quad . \quad (1)$$

termed the *exponential law of error*. In this equation k and h are constants, and e is the base of the Napierian system of logarithms—namely, 2·71828 . . .*

Although, in practice, very large accidental errors never occur, yet in the theoretical representation of the case shown by Fig. 1 errors of all magnitudes up to $\pm \infty$ are held possible, since the X-axis is an asymptote.

The probability of an error lying between the two positive magnitudes O B and O A is represented by the ratio of the area A B C D to the total area between the curve and the X-axis. Similarly, the probability that an error will be smaller in magnitude than ± O A is measured by the area E F Y D A divided by the total area, O E being equal to O A.

* The fact that errors are met with in surveying, following laws other than the exponential, is remarked on in Chapter I., p. 6.

4. **The Probable Error.**—If the area E F Y D A were found to be exactly one-half of the total area, the error ± O A would be termed the *probable error*. The probability of an error being less than the "probable error" is therefore one-half (§ 1). The probability of an error being greater than the "probable error" is also one-half.

Hence the "probable error" is such that, in a large series of errors, there are as many of greater magnitude as there are of smaller magnitude. In other words, "the odds are even" that an error taken at random from the series will be greater than, or will be less than, the probable error.

To state that an angle has been ascertained as 51° 20′ 30″ with a probable error of ± 3 seconds means that it is equally likely that the true value of the angle lies between 51° 20′ 27″ and 51° 20′ 33″ as that it lies outside those limits.

5. **The Mean-Square Error.**—Another criterion of the degree of uncertainty of the result of a determination is the *mean-square error*. It is defined as the square root of the arithmetic mean of the squares of the individual errors.

Let Δ_1, Δ_2, Δ_3, etc., represent the errors committed in a series of n equally trustworthy observations, and ε_{ms} the mean-square error, then :—

$$\varepsilon_{ms} = \pm \sqrt{\left\{\frac{\Delta_1^2 + \Delta_2^2 + \Delta_3^2, \text{etc.}}{n}\right\}} = \pm \sqrt{\frac{\Sigma(\Delta^2)}{n}}. \qquad (2)$$

6. **The Average Error.**—The *average error* in a series of equally good observations of the same thing, whether a length or an angle, is defined as the arithmetic mean of the separate errors, taken all with the same sign, either plus or minus.

As it is impossible to deal with actual individual errors in a general investigation, choice has to be made of some form of mean error as a representative value. Although both the probable and mean-square errors are more commonly used for that purpose, the average error, or average deviation, as it would better be called has been selected for use throughout this book. The reasons for this choice are, firstly, that it is the standard of comparison most serviceable in a discussion of the relative accuracy of different results and processes ; secondly, because it is more readily determined in practice than either of the other two forms ; and thirdly, because it is more easily understood by those not well versed in the mathematical theory of errors.

By definition, then, the average error has the value :

$$\varepsilon_a = \pm \left\{ \frac{\Sigma(+\Delta)}{n} \right\}^{*}. \quad \quad \quad (3)$$

7. Connection between the Average, Mean-square, and Probable Errors.—It follows from theoretical considerations that, if ε_p represents the probable, ε_{ms} the mean-square, and ε_a the average error involved in the same series of observations :

$$\varepsilon_p = 0{\cdot}845\ \varepsilon_a. \quad \quad \quad (4)$$

$$\varepsilon_{ms} = 1{\cdot}253\ \varepsilon_a. \quad \quad \quad (5)$$

$$\varepsilon_p = 0{\cdot}674\ \varepsilon_{ms}. \quad \quad \quad (6)$$

$$\varepsilon_a = 1{\cdot}183\ \varepsilon_p. \quad \quad \quad (7)$$

$$\varepsilon_a = 0{\cdot}798\ \varepsilon_{ms}. \quad \quad \quad (8)$$

* The orthodox mode of expressing the average error is $\frac{\Sigma(+\Delta)}{n}$, meaning the average of the errors taken all with the positive sign, and, while the plus or minus sign is universally applied to probable and mean-square errors, it is unusual to apply it to average errors. As it is equally logical to write the average error as $\frac{\Sigma(-\Delta)}{n}$, the double sign is, however, always given to it in this book. It is, moreover, inconsistent to give average errors the *plus* sign only, and then (as is generally done) to apply the usual theory of errors to them ; since such an application involves the fundamental assumption that the error is as often *minus* as *plus*.

These results are only true when the number of observations is large, and become less and less warrantable the more the number of observations is reduced. This fact is illustrated below, by means of data quoted by Professor Crandall from Bessel's *Gradmessung in Ostpreussen.* The example also serves to show the method of deriving the mean-square and average errors from the results of actual observation.

After measuring the angles of 22 triangles in a primary triangulation, and correcting for spherical excess, Bessel found that the excess over 180° was as follows :—

No. of Triangle.	Δ.	Δ².	No. of Triangle.	Δ.	Δ².
	Seconds.			Seconds.	
1, .	+ 0·36	0·130	12, .	0·00	0·000
2, .	+ 0·93	0·865	13, .	− 1·36	1·850
3, .	− 0·51	0·260	14, .	+ 1·86	3·460
4, .	− 1·46	2·132	15, .	− 0·42	0·176
5, .	− 0·95	0·902	16, .	+ 1·68	2·822
6, .	− 1·40	1·960	17, .	+ 1·62	2·624
7, .	+ 1·76	3·098	18, .	+ 1·62	2·624
8, .	+ 0·92	0·846	19, .	+ 1·67	2·789
9, .	+ 0·56	0·314	20, .	− 0·72	0·518
10, .	0·00	0·000	21, .	− 1·35	1·822
11, .	− 0·59	0·348	22, .	− 0·98	0·960

$\Sigma(\Delta^2) = 30{\cdot}500$; therefore, from eq. (2)—

$$\epsilon_{ms} = \sqrt{(30{\cdot}500 \div 22)} = \pm\ 1{\cdot}18 \text{ secs.}$$

And from eq. (6)—

$$\epsilon_p = \pm\ 0{\cdot}674 \times 1{\cdot}18 = \pm\ 0{\cdot}80 \text{ sec.}$$

$\Sigma(+\ \Delta) = 22{\cdot}72$; therefore, from eq. (3)—

$$\epsilon_a = \pm\ 22{\cdot}72 \div 22 = \pm\ 1{\cdot}03 \text{ secs.}$$

If this last result be multiplied by 1·253—thus, $1{\cdot}253 \times \pm\ 1{\cdot}03 = \pm\ 1{\cdot}29$—we obtain, according to eq.

(5), another value for ε_{ms}; but it is seen that there is a discrepancy of $\pm$ 0·11 second between the two values. This is due to the limited number of the observations.

8. **Apparent and True Errors.**—It is universally accepted, both as an axiom in the theory of errors and as a commonplace by all who have to deal with observed quantities, that the best representative value of a quantity, as ascertained from a series of equally trustworthy measurements, is the arithmetic mean of the several determinations. Were it possible to make an infinite number of measurements, it is certain that their arithmetic mean would give the true value of the quantity in question. It is equally certain that the fact that the number of determinations possible in practice is limited, causes the arithmetic mean itself to be influenced by error.

After having obtained a number of results for the length of a triangulation base, say, and taken their mean, one can write down the errors affecting the measurements by subtracting the mean from each of them. Because the absolute length of the base is unknown, the errors so obtained cannot be true errors; hence they are termed *residual* or *apparent* errors. On the other hand, such errors as those stated in the table above are *true errors*, since one knows with absolute precision that the sum of the angles of a plane triangle is 180°. The surveyor, then, has to do with both true and apparent errors; the first when dealing with the summation errors of triangles or polygons, and the second when dealing with lengths.

The theory of errors allows the error of the arithmetic mean of n equally good observations, affected with apparent errors $\Delta_1, \Delta_2 \ . \ . \ .$, to be assessed :—

$$\left.\begin{array}{r}\textit{Apparent average error}\\ \textit{of arithmetic mean}\end{array}\right\} = \pm \frac{\Sigma(+\Delta)}{n\sqrt{n}} = \pm \frac{\varepsilon_a}{\sqrt{n}}, \quad . \quad (9)$$

$$\left.\begin{array}{r}\textit{Real average error of}\\ \textit{arithmetic mean}\end{array}\right\} = \pm \frac{\Sigma(+\Delta)}{n\sqrt{n-1}} = \pm \frac{\varepsilon_a}{\sqrt{n-1}}, \quad (10)$$

Inasmuch as eq (10) permits the real average error of the mean to be found from a set of residuals or apparent errors, it is particularly serviceable in surveying problems. Its use is illustrated in the following example :—

After correcting for temperature and slope, the results of six measurements of the length of a base by an ordinary steel tape were obtained as 1014·52, 1014·59, 1014·63, 1014·50, 1014·60, and 1014·58 feet, of which the arithmetic mean is 1014·57 feet. By subtracting each result from the mean, the residual errors are respectively found to be ·05, ·02, ·06, ·07, ·03, and ·01 foot, their signs being disregarded. The apparent average error, ε_a, of a single observation is obtained by taking the mean of the residuals—*i.e.*, ·04 foot. Then, from eq. (10), since $n = 6$, we have—

$$\textit{Real average error of arithmetic mean} = \pm \frac{\cdot 04}{\sqrt{5}} = \pm \cdot 018 \text{ ft.}$$

It is usual to express such an error as a fraction of the total length of the line, thus :—

$$\textit{Average fractional error} = \pm \frac{\cdot 018}{1{,}015}, \text{ or about 1 in 56,300.}$$

Having regard to the fact that the discrepancies between the six readings above are, if anything, greater than those one usually expects in practice, we may safely conclude that when the mean temperature is known to the nearest degree Fahrenheit, the slope determined, and the pull registered by a spring balance, an accuracy of 1 in 50,000 is not difficult to obtain with an ordinary steel tape without special tackle.*

The method just applied is useless unless the tape is standardised. If the tape is unstandardised cumulative error generally enters to an extent sufficient to over-

* Also, see Johnson and Smith's "Theory and Practice of Surveying," seventeenth edition, p. 567. (Wiley, New York.)

whelm accidental error, and to render the above mode of assessing average error nugatory. It must also be remembered that the result of such a calculation will be worthless unless a fair number of measurements are made. A base ought to be measured at least six times before the error can be analysed. The mathematician will probably say that that is far too small a number; yet the author has seen calculations for probable error based on two measurements of a line!

9. **Weight.**— Not infrequently in practice we are called on to make a comparison between, or a combination of, several determinations not equally worthy of trust, and then we have to estimate the relative degree of confidence we can place on each determination; or, as is commonly said, we have to give to each measurement its due *weight*. For example, if a line were measured once by a steel tape and once by a chain, giving respectively 280·5 and 279 feet as its length, it would hardly be fair to take the simple arithmetic mean of these figures as the distance in question, since it is certain that the first measurement is more true than the second. From a consideration of the circumstances we may decide that the first measurement is worth say twice as much as the second, and a better mean would then be obtained by adding 280·5, 280·5, and 279, and dividing by 3, instead of adding 280·5 and 279, and dividing by 2. We should then say that the tape measurement had a weight of 2 as compared with the chain measurement of unit weight, and the result would be termed the *weighted mean*.

Putting this in the form of a general expression, let a_1, a_2, a_3 . . . be a series of measurements of the same thing, and let p_1, p_2, p_3 . . . be their respective weights. Then their weighted mean (η) will be :—

$$\eta = \frac{a_1 p_1 + a_2 p_2 + a_3 p_3 \ldots}{p_1 + p_2 + p_3 \ldots} \qquad . \quad . \quad (11)$$

It is evident that the arithmetic mean forms a particular case of this more general average, and is only true when the weight of each of the measurements involved is unity.

It is known from theory that *the weight of a measurement is inversely as the square of its average error.* The following example is inserted to show how this fact may be utilised :—

Two observers, A and B, measure a triangulation angle. A obtains 65° 21′ 30″ as its value, and B obtains 65° 21′ 10″. A's experience in triangulation, when analysed by the process which was applied to Bessel's results in § 7, shows his average error to be ± 10″, while B's is ± 20″ for this class of work. What is the most probable value of the angle ?

According to the above theorem—

$$\textit{Weight of A's observation} : \textit{Weight of B's observation} :: 1 \div 10^2 : 1 \div 20^2,$$

$$\textit{i.e.,} \quad :: \quad 4 \quad : \quad 1.$$

Then, applying eq. (11), we have :—

$$\textit{Weighted mean of the results} = \frac{4 \times 65^\circ\, 21'\, 30'' + 65^\circ\, 21'\, 10''}{5}$$

$$= 65^\circ\, 21'\, 26''.$$

10. **Indirect Observations.**—Many results in surveying are not obtained from direct measurement, but indirectly from other measurements. In trigonometrical levelling, for example, the quantities actually observed are the slope-distance and the angle of elevation or depression, the difference of elevation being calculated by multiplying the former by the sine of the latter. It is, therefore, necessary to inquire how error is transmitted from the actual observations to the quantities dependent on them.

If I_1, I_2, I_3 . . . represent a series of independent quantities respectively affected by the average errors

$\pm\varepsilon_1, \ \pm\varepsilon_2, \ \pm\varepsilon_3 \ . \ . \ .$ and x is a quantity connected to them by the general relation ·—

$$x = f(I_1, I_2, I_3 \ . \ . \ . \ ). \qquad (12)$$

It can be shown that the average error in x (ε_x) is expressed by—

$$\varepsilon_x = \pm\sqrt{\left\{\left(\frac{\delta f}{\delta I_1}\right)^2 \varepsilon_1^2 + \left(\frac{\delta f}{\delta I_2}\right)^2 \varepsilon_2^2 + \left(\frac{\delta f}{\delta I_3}\right)^2 \varepsilon_3^2 \ . \ . \ . \right\}}. \qquad (13)$$

The four following articles give instances of the application of this most important equation.

11. **Average Error affecting a Sum.**—Let eq. (12) take the form :

$$x = I_1 + I_2 + I_3 \ . \ . \ . \qquad (14)$$

In this case, $\dfrac{\delta f}{\delta I_1} = \dfrac{\delta f}{\delta I_2} = \dfrac{\delta f}{\delta I_3} \ . \ . \ . = 1,$

and eq. (13) becomes—

$$\varepsilon_x = \pm\sqrt{\{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 \ . \ . \ . \}}. \qquad (15)$$

That is to say, the average error affecting the sum of a number of quantities is equal to the square root of the sum of the squares of the average errors of those quantities.

This valuable theorem can be used in the reverse manner in finding the average error of a single measurement when the average error of the sum of several such is known.

(*a*) For example, if we know that the individual angles of a triangle were measured with the same care —or, in other words, that they have equal weight— it is evident that their average errors are alike. The average error (ε_x) in summation of the angles of a plane triangle will, therefore, be $\pm\sqrt{(e^2 + e^2 + e^2)}$, where e is the average error of a single angle. Otherwise expressed—

$$\varepsilon_x = \pm e\sqrt{3}, \text{ or } e = \pm \varepsilon_x \div \sqrt{3}. \qquad (16)$$

From this equation we learn that the average error in a single angle of the triangles discussed in § 7 was—

$$\pm 1{\cdot}03 \div \sqrt{3}, \text{ or } \pm 0{\cdot}59 \text{ second.}$$

(*b*) In the method of angular measurement known as *reiteration*, a triangulation angle is recorded two or more times by using different parts of the graduated limb of a theodolite, and the arithmetic mean of the results taken. With a transit theodolite it is usual to reverse the telescope and make the measurements over again. This is termed reiteration on both "faces"—*i.e.*, "face-left" and "face-right" Thus with three reiterations on each face the angle is turned-off six times.

If it were found, after considerable experience in minor triangulation, using, say, a reliable 5-inch transit theodolite and employing three reiterations on each face, that the average error of summation of the three angles of a triangle was $\pm$ 12 seconds, the average error of each angle would be $\pm 12 \div \sqrt{3}$, or $\pm$ 7 seconds. As each angle is the mean of six measurements of equal weight, the average error of one measurement would be $\pm 7\sqrt{6}$, or $\pm$ 17 seconds.

Since the average error is inversely proportional to the square root of the number of reiterations, and since weight is inversely as the square of the average error (§ 9), it follows that the weight of an angle measured by reiteration is directly as the number of reiterations.

(*c*) Occasionally it happens that the angles of a triangle have not the same weight, and the question arises, What is the average error of summation of such angles?

Assuming that we found the average error of an angle measured by three reiterations on each face to be $\pm$ 7 seconds, what is the average summation error of a triangle in which one angle is measured by three, another by two, and the third by one reiteration? By one reiteration a simple face-left and face-right determination is meant.

Making use of the results obtained in the preceding section :—

The first angle has a weight of 3, and an average error of $\pm$ 7 secs.

,, second	,, 2,	,, ,,	$\pm 7\sqrt{\frac{3}{2}}$.
,, third	,, 1,	,, ,,	$\pm 7\sqrt{3}$.

Therefore, by eq. (15)—

$$\left.\begin{array}{r}\textit{Average summation}\\ \textit{error of the triangle}\end{array}\right\} = \pm\sqrt{\left\{(7)^2+\left(7\sqrt{\frac{3}{2}}\right)^2+(7\sqrt{3})^2\right\}}$$

$$= \pm\ 16{\cdot}4 \text{ secs.}$$

(*d*) Two angles only of a triangle were measured, the third being obtained by difference. Experience with the instrument employed led to the conclusion that the average error of each of the measured angles was $\pm$ 7 seconds. What was the average error and relative weight of the third angle ?

Let x_1, x_2, and x_3 be the three angles in question, and ϵ_1, ϵ_2, and ϵ_3 their respective average errors.

$$x_1 + x_2 \equiv 180^\circ - x_3.$$

By eq. (15) $\qquad \pm\sqrt{(\epsilon_1^2 + \epsilon_2^2)} = \epsilon_3,$

or $\qquad \epsilon_3 = \pm\sqrt{(98)} = \pm\ 10$ secs., very nearly.

The weight is inversely as the square of the average error ; so that—

Weight of x_3 : *weight of* x_1 *or* x_2 :: 49 : 98
:: 1 : 2.

12. **Average Error affecting a Difference.**—Let eq. (12) take the form :

$$x = \mathrm{I}_1 - \mathrm{I}_2. \qquad (17)$$

Then $\qquad \dfrac{\delta f}{\delta \mathrm{I}_1} = 1;\ \dfrac{\delta f}{\delta \mathrm{I}_2} = -\ 1,$

and eq. (13) becomes—

$$\varepsilon_x = \pm \sqrt{\{\varepsilon_1^2 + \varepsilon_2^2\}} . \quad . \quad . \quad (18)$$

Or, the average error affecting the difference of two quantities is equal to the square root of the *sum* of the squares of the average errors affecting those two quantities.

13. **Average Error affecting a Product.**

$$\text{Let } x = I_1 I_2 I_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\frac{\delta f}{\delta I_1} = I_2 I_3 ; \quad \frac{\delta f}{\delta I_2} = I_1 I_3 ; \quad \frac{\delta f}{\delta I_3} = I_1 I_2 ;$$

therefore, $\varepsilon_x = \pm \sqrt{\{(I_2 I_3 \varepsilon_1)^2 + (I_1 I_3 \varepsilon_2)^2 + (I_1 I_2 \varepsilon_3)^2\}}$. (20)

Eq. (13) permits of an examination into the errors affecting the process of trigonometrical levelling. We will discuss the case as it occurs in ordinary underground or surface practice, where the lines are rarely long enough to make a correction for refraction necessary, and where the vertical angles are measured both face-left and face-right to eliminate cumulative error.

The reduction formula in trigonometrical levelling is

$$H = L \sin \theta, \quad . \quad . \quad . \quad (21)$$

where H is the difference of elevation of the centre of the theodolite and the point sighted, L the inclined distance between those points, and θ the angle of slope of the line joining them.

Here we deal for the first time with a product (L sin θ) involving a trigonometric function.

Let the average error of the length, L, be ε_L, that of the angle be ε_θ, and that of H, ε_H.

$$\frac{\delta f}{\delta L} = \sin \theta ; \quad \frac{\delta f}{\delta \theta} = L \cos \theta ;$$

therefore by eq. (13) $\varepsilon_H = \pm \sqrt{\{\varepsilon_L^2 \sin^2\theta + \varepsilon_\theta^2 L^2 \cos^2 \theta\}}$. (22)

Now, it is generally conceded that the error in linear measurement is proportional to the square root of the distance (see § 28), or—

$$\varepsilon_L = K\sqrt{L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

K being a constant. Eq. (22) may, therefore, be written—

$$\varepsilon_H = \pm\sqrt{\{K^2 L \sin^2\theta + \varepsilon_\theta^2 L^2 \cos^2\theta\}}. \quad . \quad . \quad (24)$$

The commonest way of expressing error in levelling is as a fraction of the horizontal distance covered.* In this case the horizontal distance equals L cos θ, and, therefore,

$$\frac{\varepsilon_H}{L\cos\theta} = \pm\sqrt{\left\{\frac{K^2}{L}\tan^2\theta + \varepsilon_\theta^2\right\}}. \quad . \quad . \quad (25)$$

This is a minimum when $\theta = 0°$; increases rapidly as θ increases ; and diminishes as L increases.

From this analysis we learn that the error per mile in trigonometrical levelling is least when the individual lines are as long as possible, and when steep sights are avoided—a conclusion which agrees exactly with that derived from experience.

14. Average Error affecting a Quotient.

Let
$$x = \frac{I_1}{I_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\frac{\delta f}{\delta I_1} = \frac{1}{I_2}; \quad \frac{\delta f}{\delta I_2} = -\frac{I_1}{I_2^2};$$

therefore,
$$\varepsilon_x = \pm\sqrt{\left\{\frac{\varepsilon_1^2}{I_2^2} + \frac{\varepsilon_2^2}{I_2^2}\left(\frac{I_1}{I_2}\right)^2\right\}}. \quad . \quad . \quad (27)$$

15. **Vector Errors.**—So far, all the errors dealt with have been scalar quantities ; the reader is now asked to conceive of *vector errors*.

"Surveying is the art of making measurements which determine the *relative* position of two or more points." But we cannot consider the position of a point to be determined

* See (*b*) of footnote, p. 9.

with relation to another, which we will term the origin, until we know exactly how far it is from all of three planes meeting at the origin. These planes are conveniently taken as the vertical plane of the meridian, the vertical plane normal to the meridian, and the horizontal plane containing the origin respectively. In other words, we require to know three co-ordinate distances. Now, in ascertaining these co-ordinates, error is sure to arise in each. Such errors have in common the property of disturbing the location of the point in space, but they differ from each other in their planes of action.

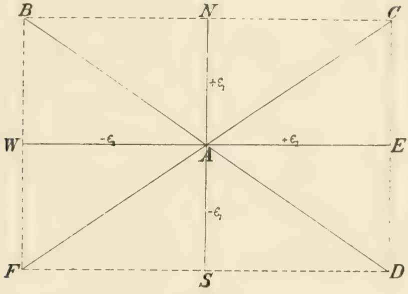

Fig. 2.

Independent errors influencing the position of the same point, but acting in different directions, are termed vector errors.

It has now to be seen how vector errors may be combined or added, in order to find their resultant effect.

16. **Summation of Two Vector Errors acting at Right Angles.**—Leaving levelling errors out of the question for the moment, let us consider the total effect of two vector errors, the one causing uncertainty in the position of a point A (Fig. 2) in the north-and-south direction, and the other in the east-and-west direction. Let their average

magnitudes be respectively $\pm \varepsilon_1$ and $\pm \varepsilon_2$, and consider them to be shown to scale in Fig. 2. Under the influence of ε_1 only, A would be shifted to N or to S; while if ε_2 were to act alone, A's average movement would be to E or to W. It is evident from the figure that no matter with what combination of signs the vector errors act together (whether $+\varepsilon_1$ with $+\varepsilon_2$, $+\varepsilon_1$ with $-\varepsilon_2$, $-\varepsilon_1$ with $+\varepsilon_2$, or $-\varepsilon_1$ with $-\varepsilon_2$), the resultant movement of the point (to C or B or D or E, as the case may be) always has the magnitude $\sqrt{(\varepsilon_1^2 + \varepsilon_2^2)}$. Or, if R be the average sum of two vector errors ε_1 and ε_2 acting at right angles:

$$R = \sqrt{(\varepsilon_1^2 + \varepsilon_2^2)}. \quad . \quad . \quad . \quad (28)$$

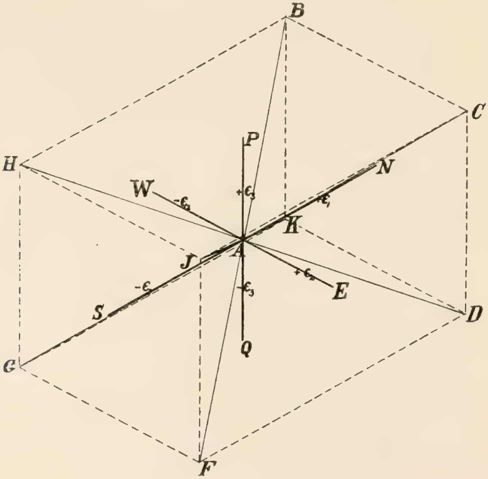

Fig. 3.

17. Summation of Three Vector Errors acting mutually at Right Angles.—Adopting a similar graphical method of representation, this case is shown isometrically by Fig. 3. If A were under the action of ε_1 alone, it would have an average deviation to N or S; if under the action of ε_2 alone, it would move to E or W; if under the action of ε_3 alone, it would have an average shift upwards to P or downwards to Q.

B C J H K D F G is a figure bounded by vertical and horizontal planes, and having the points N, S, E, W, P, and Q each in the centre of one of its sides. By reasoning identical with that used in the last paragraph, the average movement of A under the combined influence of the vector errors must be to one of the eight corners B, C, J, H, K, D, F, or G, and no other case can be conceived. But as A is in the centre of the figure, the magnitude, R, of the average resultant deviation is the same in all the eight cases, and is equal to $\sqrt{(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)}$. or—

$$R = \sqrt{(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)}. \quad . \quad . \quad . \quad (29)$$

18. **Summation of any Number of Vector Errors acting at any Angle and in any Plane.**—Let the position of the point A (Fig. 4) be affected by independent vector

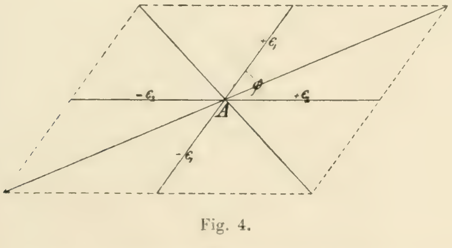

Fig. 4.

errors, whose average magnitudes are respectively $\pm\, \varepsilon_1$ and $\pm\, \varepsilon_2$, and let ϕ be the angle between their respective directions of action.

The diagram shows that the square of the average sum of these errors is either

$$R^2 = \varepsilon_1^2 + \varepsilon_2^2 + 2\,\varepsilon_1\,\varepsilon_2 \cos\phi,$$

or

$$R^2 = \varepsilon_1^2 + \varepsilon_2^2 - 2\,\varepsilon_1\,\varepsilon_2 \cos\phi.$$

Now, if we were to examine a considerable number of instances in which these vector errors influence the position of A, and were to take their average resultant effect, we should find that the mean value of R^2 would come very nearly equal to $\varepsilon_1^2 + \varepsilon_2^2$, since the terms in $\cos\phi$, being as often negative as positive, would only affect that mean value to an extent negligible in com-

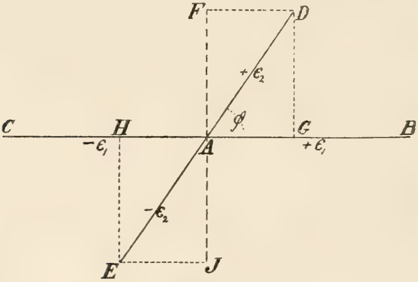

Fig. 5.

parison with the other two terms. Hence we may write—

$$R = \sqrt{(\varepsilon_1^2 + \varepsilon_2^2)}.$$

It may likewise be shown that the average sum of n vector errors simultaneously influencing the position of a point in space is—

$$R = \sqrt{(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 \; . \; . \; . \; \varepsilon_n^2)}, \quad . \quad . \quad (30)$$

and that it is immaterial to the result whether their directions lie in the same plane or not.

Alternative Proof.—Let the magnitudes and clinures of two average vector errors $\pm\,\varepsilon_1$ and $\pm\,\varepsilon_2$ be represented by A B, A C, and A D, A E (Fig. 5). Now, it has been shown in § 16 that two vector errors at right angles, such as A F, A J and A G, A H, would, in combination, have the same effect as $\pm \varepsilon_2$, since

$$AD^2 = AF^2 + AG^2 \text{ and } AE^2 = AJ^2 + AH^2.$$

In other words, it is permissible to consider the influence of $\pm \varepsilon_1$ as being due to two components at right angles—namely, A F, A J and A G, A H. The total effect of $\pm \varepsilon_1$ and $\pm \varepsilon_2$ on the point, then, will be equivalent to the combined influences of AB, AC, A F, A J, and A G, A H The summation of A B, A C and A G, A H, being scalar quantities, presents no difficulty ; it equals $\pm \sqrt{(AB^2 + AG^2)}$ by § 11. The total effect on the point must, therefore, be equivalent to the sum of the vectors $\pm$ A F and $\pm \sqrt{(AB^2 + AG^2)}$ acting at right angles ; and this, by § 16, is—

$$R = \sqrt{(AF^2 + AB^2 + AG^2)} = \sqrt{(AB^2 + AD^2)} = \sqrt{(\varepsilon_1{}^2 + \varepsilon_2{}^2)},$$

or the same result as that obtained above.

We are now able to state in general terms the following theorem :—

The average magnitude of the error in position of a point influenced by two or more vector errors is equal to the square root of the sum of the squares of the average magnitudes of the vector errors, and is independent of their relative clinures.

It will be observed that this is a generalisation of the well-known principle represented by eq. (15).

The importance of the theorem to the present work cannot be exaggerated. Very many of the conclusions adduced in the following chapters depend upon it. Indeed, it facilitates so considerably the discussion of the propagation of error in surveys that without it this book could not have been written.

The following example is inserted to illustrate the method of adding vector errors :—

A and B (Fig. 6) are two fixed points, whose relative positions have been determined by some process considerably superior to ordinary chaining. To ascertain the position of C with regard to them, the lines a and b are chained in order that C may be fixed on the plan by the usual method of intersecting arcs. By how much, on the average, will errors in chaining disturb the location of C ?

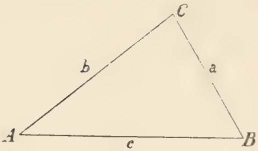

Fig. 6.

Let $\pm a_1$ and $\pm b_1$ represent respectively the average errors in measuring a and b. The ascertained position of C will be affected by both these errors. They are vector errors, since they act in different directions ; therefore, by the theorem stated above, their combined average magnitude is $\sqrt{a_1^2 + b_1^2}$. Assuming the average error in linear measurement to be proportional to the square root of the measured distance, we have $a_1 = K\sqrt{a}$ and $b_1 = K\sqrt{b}$, or—

$$\textit{Average disturbance of } C = K\sqrt{a + b}.$$

The result, as stated, does not include the errors due to plotting.

19. **Rejection of Doubtful Observations.**—To turn to a rather different matter, we may inquire if the theory of errors gives any assistance to an observer endeavouring to decide whether a certain measurement at variance with others of the same length or angle ought to be rejected or not.

For example, ought the seventh observation (viz., 72° 14′ 30″) to be rejected from the following series of measurements of an angle before taking the mean?

72° 15′ 20″	72° 14′ 30″
15′ 20″	15′ 00″
15′ 00″	15′ 20″
15′ 15″	15′ 30″
15′ 30″	15′ 30″
15′ 30″	15′ 15″

No one is so well qualified to settle the question as the observer himself. He ought first to ask himself if there was any circumstance in making the measurement in question which would lead him to consider it less trustworthy than the others; for, if so, it may be deleted. We here see the necessity of noting in the "remarks" column of the field book, while the measurements are proceeding, any occurrence which may throw doubt upon an observation. Above all, the observer must rid his mind of all bias not based on evidence, and of all desire to cancel a measurement as soon as it is made merely because it is not quite in agreement with those which have gone before. "It is in the matter of the rejection of doubtful or discordant observations," says Prof. Holman, "that an observer's integrity in scientific or technical work meets its first test."*

When an examination of the field-work fails to reveal any sound reason for rejecting a discordant observation, it is useful to have a criterion to guide the observer in settling whether to let the measurement stand or not. A considerable number of such criteria have been advanced by mathematicians, most of which suffer from the bad drawback of being complicated. The following rule is to be recommended on account of its great simplicity:—

* S. J. Holman, *Discussion of the Precision of Measurements.* New York. Second edition, 1901.

Take the mean of all the observations without exception. From this mean obtain by subtraction the residual or apparent errors of the observations, and thence the average error of a single observation. If the apparent error of the doubtful measurement exceeds three times the average error reject it; otherwise let it stand.

Applying this criterion to the angles stated above, we find their arithmetic mean to be 72° 15′ 15″, and the residuals to be respectively 5, 5, 15, 0, 15, 15, 45, 15, 5, 15, 15, and 0 seconds, giving an average error in a single observation of ± 12·5 seconds. As 45 is greater than thrice the average error, the criterion calls for the rejection of the reading 72° 14′ 30″. After cancelling that reading, the mean value of the angle becomes 72° 15′ 19″.

20. **A Criterion of Negligibility.**—Expressions of the type $z = \pm\sqrt{x^2 + y^2}$ occur with considerable frequency when dealing with errors, and occasionally it is found that one of the terms under the radical has a very small influence in comparison with the other. The question arises, How small must y be compared with x to allow it to be considered as negligible in its influence on z?

The following criterion of negligibility is made use of now and then in subsequent chapters:—*When adding two average errors, of which the first is of greater magnitude than the second, the second may be held negligible when it is equal to or less than one-third of the first.*

If y in the above expression is one-third of x, $z = \pm\sqrt{\frac{10x^2}{9}}$, or $\pm$ 1·05x—that is to say, y's influence is only about 5 per cent. of that of x. Having regard to the fact that one can very seldom indeed assess the value of an average error to within 5 per cent., y in these circumstances may be neglected, and z considered equal to $\pm x$.

CHAPTER III.

THE BEST SHAPE OF TRIANGLES.

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21. **The Meaning of Best Shape.**—Triangles, as used in surveying, serve a variety of purposes. For example, in a trigonometrical survey a triangle is an agent for transmitting distance, in the measurement of areas it is valuable as the only rectilinear figure enclosing area whose shape is altogether known from the lengths of its sides, while in that branch of mine surveying termed shaft-connection a triangle becomes an agent for transmitting bearing. But for whatever purpose a triangle is utilised, that purpose is only served after a calculation has been made, based on some trigonometrical property of the figure ; and, in order to make such a calculation, data must be furnished from angular and linear measurements.

In attempting to define the best shape of a triangle, it is necessary to distinguish between (1) a triangle in which the unknown quantity to be computed is an angle, and (2) one from which a length or area is to be calculated.

The best shape of a triangle in which the unknown quantity is an angle, is that shape which ensures that errors in the linear and angular measurements have least influence on the angle to be calculated.

The best shape of a triangle from which a length or area is to be determined is that shape which ensures that errors in the linear and angular measurement have a minimum *proportional* influence on the calculated result.

To strive after the best shape is admittedly good practice, and the object of this chapter is to explain by means of examples a method which allows the best shape to be ascertained under any given conditions. Occasionally the results as attained by this method need some modification from what may be termed "side considerations" of a practical nature, but such modifications are seldom of sufficient moment to affect seriously the value of the theoretical conclusions.

22. **The Most Economical Shape.**—In some instances the best-shape rule is found incapable of making a full specification of the *most suitable* triangle, and another criterion—namely, that of the *most economical shape*, has to be called in as auxiliary. The most economical shape is that in which the desired result is secured with the minimum expenditure of labour, and, therefore, at a minimum cost.

Not infrequently the two criteria are at variance. In triangulation, for example, if we were only to look to the smallest expenditure of labour in fixing the relative positions of two distant points, we should elongate the triangles as much as possible, in order to cover the ground as speedily as possible. Triangulation is an instance where, in practice, the "most economical shape" is made to give way to the theoretical "best shape"—namely, the equilateral. Sometimes, however, the consensus of practical opinion is in favour of the most economical as against the best shape, and an instance of this is to be found in the simple method of chain surveying so frequently employed in determining small areas. The reader may, after he has studied this chapter, take as an exercise to prove that, of all isosceles triangles standing on a given base, the one having its equal sides infinitely long is that in which errors in measuring the sides have the least proportional influence on the calculated area. That is to say, if a chain survey is conducted purely or primarily as the means of finding the area of a piece of ground, an attenuated isosceles triangle standing on a certain base-line (if such could be arranged) would be likely to give a more accurate result than, say, a chain of equilateral triangles starting from the same base-line. Yet, of all triangles, the equilateral is that enclosing a maximum area for a given perimeter; or, in other words, the equilateral is the most economical shape in chain surveying; and every text-book tells us it is the best, without reference to its purpose.

23. **The Case of a Triangle in which two Sides and the Included Angle are measured for the Purpose of ascertaining its Area.**—Let A B C (Fig. 6) be the triangle, in which the sides b and c, together with the angle A, are measured. Let $\pm b_1$, $\pm c_1$, and $\pm A_1$ be the average errors affecting these measurements, respectively.

The formula for the area (x) is :—

$$x = \tfrac{1}{2}\, b\, c \sin A. \qquad . \qquad . \qquad . \quad (31)$$

By eq. (13) the average error ($\pm x_1$) in the calculated area is :—

$$x_1 = \pm\sqrt{\{(\tfrac{1}{2} c b_1 \sin A)^2 + (\tfrac{1}{2} b c_1 \sin A)^2 + (\tfrac{1}{2} b c A_1 \cos A)^2\}}. \quad (32)$$

By dividing eq. (32) by eq. (31) the *average fractional error* in area is obtained :—

$$\frac{x_1}{x} = \pm\sqrt{\left\{\left(\frac{b_1}{b}\right)^2 + \left(\frac{c_1}{c}\right)^2 + A_1^2 \cot^2 A\right\}}. \quad . \quad (33)$$

If the accuracy of the measurements is known, this equation allows the average error in area—expressed as a fraction of the area—to be computed. The triangle best serves its function when $\frac{x_1}{x}$ is a minimum, and it is seen that this is secured (so far as the effect of the angle is concerned) when A is a right angle.

If the error in linear measurement were directly proportional to the length measured, both $\frac{b_1}{b}$ and $\frac{c_1}{c}$ would be constants, and, so far as the effects of the sides are concerned, right-angled triangles would be equally good no matter what the lengths of the sides happened to be. But we know that the error in linear measurement is, for a case of this kind, actually more nearly proportional to the square-root of the length, or $b_1 = K\sqrt{b}$ and $c_1 = K\sqrt{c}$, where K is a constant. Eq. (33) now reduces to—

$$\frac{x_1}{x} = \pm \sqrt{\left\{ K^2\left(\frac{1}{b} + \frac{1}{c}\right) + A_1^2 \cot^2 A \right\}}, \quad . \quad (34)$$

from which it is seen that the fractional error in area is smaller when the sides are long than when they are short.

Eq. (34) does not guide us as to the *relative* lengths of the measured sides. It can, however, be easily proved that when the triangle is right-angled and of a certain area, $(a + b)$ is a minimum when $a = b$.

We are now in a position to state (1) that to gain maximum accuracy in the area determination the measured angle should be as nearly a right angle as possible, and the measured sides as long as possible: and (2) that the *isosceles* right-angled form is the most economical shape, inasmuch as it ensures the smallest total amount of chaining for a given area.

24. **The Case of a Triangulation Triangle.**—To solve any triangle of a triangulation system it is necessary to know the three angles and the length of one side. In the first triangle of the survey the length of one side is actually measured, and this line is termed the *base* of the system. Every angle in every main triangle is measured by the theodolite: it is only in triangles of secondary importance (secondary or tertiary triangles) that one is permitted, now and then, to measure two angles only, leaving the third to be obtained by difference.

The errors of measurement in triangulation are of two kinds—namely, (*a*) errors in the angles, and (*b*) error in the base. Now, it will be subsequently shown (§ 32) that if the angles of a triangle have been measured by the same method and observer, and with the same care, the average angular error may be taken as of equal amount in each of them. In order to conform with the notation used in later chapters, the letter v will be taken in this section as representing the average error in measuring an angle, the error being expressed in radians unless otherwise stated. A uniform method is adopted of representing linear errors: thus $\pm a_1$ is the average error in a side a of a triangle, $\pm c_1$ in a side c, and so on.

We need not be concerned with the question of spherical excess other than to say, that if the triangle is so large that the excess is appreciable, it must be corrected before the following method and results are applicable.

Consider the case of a single triangle standing on a measured base—such a triangle, for example, as A B C. Fig. 6. The triangle will usually form one link in a chain of triangles; hence errors in measuring the angles or base will not only affect the calculated lengths of the unmeasured sides, but will be carried forward through the whole scheme; it is, therefore, necessary to deal with the triangle as *an agent for transmitting distance.*

A chain of triangles may subsequently be built upon the side a, or upon b, or upon both. In a general discussion, therefore, these sides must be considered as being of equal importance, in other words, they must be given an equal "weight." Hence, in determining the *best shape* of a triangle with c as base, a must not be allowed to suffer in accuracy for the sake of b, nor b for the sake of a. This condition can only be realised when a and b are kept equal in length—that is to say, the best-conditioned triangle must be of isosceles shape.

Now, a triangle will have the best shape when it fulfils its function as a distance-transmitter with a minimum of error. Thus A B C (Fig. 6) will be of the best shape when the error in a (or in b) forms the smallest possible proportion of the length of a (or of b)—*i.e.*, when $\frac{a_1}{a} = \frac{b_1}{b}$ is a minimum—for it is the fractional error $\frac{a_1}{a}$ or $\frac{b_1}{b}$ which must be considered in its effect on the next triangle of the system.

The triangle A B C being isosceles when of the best shape, we have—

$$C = 180° - 2\,A. \qquad (35)$$

To calculate the length of the side a, the "sine rule" is used, thus—

$$a = \frac{c \sin A}{\sin C}. \qquad (36)$$

By eq. (13)—

$$a_1 = \pm\sqrt{\left\{\left(\frac{\delta a}{\delta c}\right)^2 c_1^2 + \left(\frac{\delta a}{\delta A}\right)^2 v^2 + \left(\frac{\delta a}{\delta C}\right)^2 v^2\right\}}$$

$$= \pm\sqrt{\left\{\left(\frac{\sin A}{\sin C}\cdot c_1\right)^2 + \frac{c^2 v^2 \cos^2 A}{\sin^2 C} + \frac{c^2 v^2 \sin^2 A \cos^2 C}{\sin^4 C}\right\}}$$

$$= \pm\sqrt{\left\{\frac{\sin^2 A}{\sin^2 C}\cdot c_1^2 + a^2 v^2 \cot^2 A + a^2 v^2 \cot^2 C\right\}}.$$

Therefore, the average fractional error in a is determined by—

$$\frac{a_1}{a} = \pm\sqrt{\left\{v^2(\cot^2 A + \cot^2 C) + \left(\frac{c_1}{c}\right)^2\right\}}. \qquad (37)$$

It is evident, then, that the fractional error in a can be made to alter by varying the shape of the triangle, keeping the length of the base, c, and the fractional error in the base, $\frac{c_1}{c}$, constant. Substituting from eq. (35) we obtain, for an isosceles triangle—

$$\frac{a_1}{a} = \pm\sqrt{\left\{v^2(\cot^2 A + \cot^2 2A) + \left(\frac{c_1}{c}\right)^2\right\}}, \qquad (38)$$

and this is a minimum when $(\cot^2 A + \cot^2 2\,A)$ is a minimum. By differentiating the latter expression with respect to A, and equating the result to zero, it is found that A = 56° 15′ gives (38) a minimum value.

That is to say, the theoretically perfect triangle for triangulation purposes is an isosceles one, having the angles at the base 56° 15′ and that at the apex 67° 30′.

Were a triangulation survey to be laid out composed entirely of triangles of this perfect shape, the lengths of the sides would gradually decrease as one proceeded further and further from the base. Now this would not be desirable, since, as the sides shorten, centring errors (see § 32) would in time cease to have a negligible influence, and the ultimate effect would thus be to increase the average error in the angles. Again, with a system in which the triangles were gradually being reduced in size, more triangles would be needed to cover a given area of ground. Therefore, although the result last deduced possesses a considerable theoretical interest, there is no need to differ from the opinion generally held, to the effect that the best shape of triangle for practical purposes is the equilateral.

Attempts are sometimes made to prove the equilateral to be the best-conditioned triangle by a graphical method ; the premises, however, are generally defective, and in some cases quite unreal. The author believes the problem to be too intricate to be treated adequately by any graphical mode.

The curves (Fig. 7) are constructed from equation (38) ; they show the relation between the average fractional error, $\frac{a_1}{a}$, in the side, a, of an isosceles triangle, A B C, in which the apical angle, C, assumes all values between 0° and 180°. For curve A the fractional error in the base —namely, $\frac{c_1}{c}$, is assumed negligible—for curve B it is taken as $\pm 1 \div 20{,}000$, or 5×10^{-5}, and for curve C as $\pm 1 \div 5{,}000$, or 2×10^{-4}.

In all three cases the average error, v, in angle, is taken as $\pm$ 10 seconds ($4{\cdot}85 \times 10^{-5}$ radians).

These curves supply a considerable amount of information of practical importance. They provide, first of

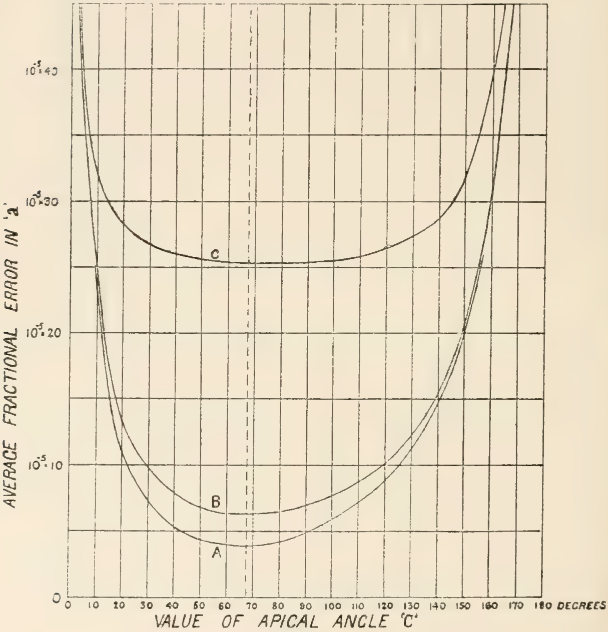

Fig. 7.

all, a further justification in taking the equilateral as the perfect shape, the difference in $\frac{a_1}{a}$ being inconsiderable as between C = 67° 30′ and C = 60°. Secondly, curve

B, which is constructed with $\frac{c_1}{c}$ and v of similar magnitudes, illustrates how rapidly $\frac{a_1}{a}$ increases when C exceeds 120° or becomes less than 30°, but also shows that the *rate* of increase is not so rapid when C assumes unduly large as when it assumes unduly small values. As the conditions in respect of the relative accuracy of base and angles may be said to lie most usually somewhere between those represented by curves A and B, they may be taken to demonstrate the well-known rule to the effect that no important triangle should have an angle less than 30° or greater than 120°. Curve C, however, shows that when the error in base greatly exceeds that in angle, these limits may be set much further apart without any appreciable reduction in accuracy. Indeed, when $v = \pm 10$ seconds, and $\frac{c_1}{c} = \pm 1 \div 5{,}000$ the angle C may be given any value from 15° to 145° without much risk. We learn, then, that whether a triangle is to be considered permissible for triangulation purposes does not depend merely on its shape, but also on the relative accuracy of the base and angles.

Now, since the lines of a triangulation become affected with a greater average error the further one proceeds from the base (see § 41), and as each of those lines virtually serves as base for the triangle depending on it, it follows that it is permissible in practice to use triangles towards the end of a system departing more from the perfect shape than those near the beginning. We can also conclude that, whenever circumstances are such as to preclude accurate base-measurement, the best aim in laying out the scheme is not to arrange the triangles as nearly equilateral as possible, but purposely to elongate them so as to cover the area with as *few* triangles as possible—in other words, to aim for the most economical rather than for the theoretically best shape.

Referring again to curves A and B, it is seen that even when the base is measured with an accuracy equal or superior to that of the angles, too great a stress may be laid on striving after the equilateral shape, for isosceles triangles in which the apical angles lie anywhere between 50° and 90° are all almost equally well-conditioned. This fact simplifies the subsequent work, for, in Chapter VI., we can proceed to consider schemes built of equilateral triangles, knowing that the results attained will be of almost equal practical value for triangles departing pretty considerably from that shape.

When the triangle is equilateral, eq. (38) becomes

$$\frac{a_1}{a} = \pm\sqrt{\left\{\frac{2}{3}v^2 + \left(\frac{c_1}{c}\right)^2\right\}}. \qquad (39)$$

25. **The Case of Weisbach's Triangle.**—In mine surveying, it is not infrequently necessary to convey true bearing underground through the medium of a single vertical shaft. To do this the usual practice is to hang two loaded wires from top to bottom of the shaft ; to find the bearing of the line joining the wires by surface observations with a theodolite, and then to use that line as a base from which to commence the underground traverse. The bearing of this base is determined at the surface either by ranging a theodolite accurately in line with the wires, or by a process of triangulation ; and the same methods are also employed at the shaft-bottom in transferring that bearing to the first lines of the underground survey. If the underground survey is extensive, very great care is needed in the operations referred to, and it is evidently a matter of considerable practical importance to ascertain to what extent errors of measurement affect the results, and to endeavour to realise the conditions in which those errors have least influence.

Let A and B (Fig. 8) represent the wires hung in the shaft, and let us assume that we have already made such surface observations as have allowed us to compute the true bearing of the line A B. Underground, we decide to transfer the bearing from A B to the traverse running into the mine by triangulation, and accordingly set the instrument at C, so that it commands a view of both wires, and also of D, the first fixed station in-bye. The shaft-connection will be complete when the bearing of C D has been ascertained from that of A B. In order to be able to solve the triangle A B C, the angle at C is carefully measured, and the lengths of the sides a, b, and c obtained by aid of the steel tape. The angles B and A are derived from the sine rule, thus :—

$$\sin B = \frac{b \sin C}{c}, \qquad (40)$$

and

$$\sin A = \frac{a \sin C}{c}. \qquad (41)$$

When these have been calculated and the angle B C D measured, it is straightforward work to get out the bearing of C D ; indeed two values for that bearing are obtainable, one from each of the two equations above, and, providing they agree closely, their mean may be taken as the result required.

Our present purpose is to find the best shape of the triangle A B C.

Let the average errors in B, C, b, and c be respectively represented thus : $\pm B_1$, $\pm C_1$, $\pm b_1$, and $\pm c_1$.

Fig. 8.

Eq. (40) may be written—

$$B = \sin^{-1}\left(\frac{b \sin C}{c}\right),$$

and, applying eq. (13), we obtain—

$$B_1 = \pm\sqrt{\left\{\left(\frac{\frac{\sin C}{c} \cdot b_1}{\sqrt{1-\left(\frac{b \sin C}{c}\right)^2}}\right)^2 + \left(\frac{\frac{b \cos C}{c} \cdot C_1}{\sqrt{1-\left(\frac{b \sin C}{c}\right)^2}}\right)^2 + \left(\frac{-\frac{b \sin C}{c^2} \cdot c_1^2}{\sqrt{1-\left(\frac{b \sin C}{c}\right)^2}}\right)^2\right\}},$$

or,

$$B_1 = \pm\sqrt{\left\{\frac{\sin^2 C\,(c^2 b_1^2 + b^2 c_1^2) + b^2\, c^2\, C_1^2 \cos^2 C}{c^4 - c^2 b^2 \sin^2 C}\right\}}. \qquad (42)$$

As the lengths are less than one tape-length, we are justified in taking $b_1 = c_1$ (see § 28); hence eq. (42) becomes—

$$B_1 = \pm\sqrt{\left\{\frac{c_1^2 \sin^2 C\,(b^2 + c^2)}{c^2(c^2 - b^2 \sin^2 C)} + \frac{b^2\, C_1^2 \cos^2 C}{c^2 - b^2 \sin^2 C}\right\}}. \quad . \quad (43).$$

The first expression under the radical measures the effect of errors in linear, and the second that of errors in angular measurement.

To ascertain what value of C will make (43) a minimum, differentiate the right-hand side with respect to C, and equate to zero. Two values will be found to satisfy the resulting equation—namely, $C = 90°$ and $C = 0°$, of which the first will make B_1 a maximum and the second a minimum. That is to say, the best shape of the triangle is attained when $C = 0°$, and the worst when $C = 90°$. Further, a glance at (43) shows that B_1 may be reduced either by increasing c or by diminishing b—in other words, the ratio $\frac{b}{c}$ ought to be as small as possible.

Making use of the criterion of negligibility of § 20, we are able to say that, so far as their influence in the calculated angle B is concerned, the errors in linear measurement will be negligible in comparison with those of angular measurement.

When $$\frac{c_1^2 \sin^2 C\,(b^2 + c^2)}{c^2\,(c^2 - b^2 \sin^2 C)} \lesseqgtr \frac{1}{9} \cdot \frac{b^2\, C_1^2 \cos^2 C}{(c^2 - b^2 \sin^2 C)},$$

i.e., when $$\tan C \lesseqgtr \frac{b\,c\,C_1}{3\,c_1} \sqrt{\frac{1}{b^2 + c^2}}. \qquad (44)$$

Considering the angle at C to be measured by a modern 5-inch transit theodolite, and the lengths by a steel tape graduated to hundredths of a foot, and taking $c = 5$ feet, $b = 10$ feet, $C_1 = \pm 5$ seconds ($2{\cdot}42 \times 10^{-5}$ radians), and $c_1 = \pm \frac{1}{200}$ foot, we obtain by substitution into eq. (44) the result—

$$C \lesseqgtr 25 \text{ minutes, about.}$$

Now the values used of the different factors have all been chosen to bring this result on the safe side ; thus c ought in practice to be more than 5 feet if the underground traverse is to be lengthy, c_1 will be less than $\pm \frac{1}{200}$ foot if average care is taken in taping, while C_1 will seldom be brought much lower than $\pm$ 5 seconds with a 5-inch theodolite, even by frequent repetitions on each face. We are, therefore, able to state without qualification that in general practice, if the angle at the theodolite is less than 25 minutes, it is unnecessary to measure the sides of the triangle with more than ordinary care, since the effect of linear errors will be negligible in comparison with that of angular error in such a triangle.

When C is less than 25 minutes, cos C is so near unity and sin C so near zero that eq. (43) reduces to—

$$B_1 = \pm \frac{b}{c} C_1. \qquad (45)$$

This result emphasises the value of making b short in comparison with c. In practice, although c is limited by the size of the shaft, one is generally afforded a wider choice as to the length of b. It is not, however, advisable to make b very short, because when C is very near A a considerable adjustment of focus is required every time the telescope is turned from A to B. This is objectionable, since one is not quite sure of variable-collimation error (due to the focussing tube not moving axially) being eliminated by taking the observations over again with the telescope reversed. Probably the most usual arrangement in practice is for the ratio $\frac{b}{c}$ to lie between 2 and 1.

Such a result as eq. (45) enables us to appreciate the value of having the theodolite telescope capable of focussing a point within 6 feet of the instrument. Thus in the Scott theodolite, as made by Messrs. Wittstock of Berlin—an excellent example of a modern instrument specially designed for mine surveys—the telescope can focus a point 3 feet from it.*

Mr. K. E. Weiss used an instrument capable of focussing down to about 2 feet,† while Prof. Uhlich went so far as to use a special theodolite in which the base was so modified that one wire could be suspended through it, and in that way he reduced b to zero.‡

The curve (Fig. 9) is constructed from eq. (43) and the following data: $b = c = 10$ feet, $C_1 = \pm 5$ seconds or $2{\cdot}42 \times 10^{-5}$ radians, $c_1 = \pm \frac{1}{200}$ foot. It shows how

* D. D. Scott, *Mine Surveying Instruments* (part 1). *Trans. Inst. Min. Engs.*, vol. xxiii., p. 616.

† K. E. Weiss, *Orientirung von Gruben-theodolitzügen durch Anschluss an zwei in einen Richtschachtgehängte Lothe*. Jahrbuch für das Berg- und Hüttenwesen im Königreiche Sachsen, 1896, Abhandlungen, p. 101.

‡ Paul Uhlich, *Kritische Betrachtungen über Lothungen in einem seigeren Schachte*, *ibid.*, p. 112. Also see D. D. Scott, *op. cit.* (part 2), *Trans. I.M.E.*, vol. xxviii., p. 660, where both these papers are referred to.

very rapidly the average error in B increases as the angle at the theodolite increases. This effect is entirely due to the influence of linear errors ; indeed, that of angular error is so slight for all values of C above 5° that it can be neglected even when C_1 is considerably more than ± 5 seconds. We may, therefore, conclude that, although

Fig. 9.

more care needs to be taken in measuring the angle than the lengths when C is less than 25 minutes, yet when it exceeds 5° this extra attention must be transferred from the angle to the sides. The curves also show how useless it is to expect good results when the triangle approaches the equilateral. The rate of increase of the average error in B is even more marked when *b* is greater than *c* ;

the error then reaches a maximum when B becomes a right angle.

This method of shaft-connection was introduced by Prof. Weisbach prior to 1850, and he was the first to show that the very narrow triangle is the best shape for the purpose.*

26. **The Case of the Broken Base.**—Sometimes, when laying-out a minor triangulation in hilly country, it is found impossible to obtain a straight base of sufficient length. In other cases, marshy ground or other obstructions prevent a straight base-line being taken in what may otherwise be a suitable position for it. Recourse is then had to a "broken" base, composed of two sections, *a* and *b* (Fig. 10), which embrace some angle other than 180° between them. Care must be taken to select the stations A and B so that they are visible, one from the

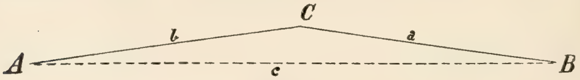

Fig. 10.

other. The three angles of the triangle are measured along with the sides *a* and *b*, and, to check the result, the virtual base, *c*, is calculated from *both* of the formulæ:—

$$c = \sqrt{a^2 + b^2 - 2\,a\,b\cos C}, \qquad (46)$$

$$\text{and } c = b\cos A + a\cos B. \qquad (47)$$

The fact that short lines are always to be carefully avoided in triangulation, owing to the large average error in angle resulting from their introduction, would lead one to make the two measured lengths as nearly equal as possible, even if the symmetry of eq. (47) did

* See L. H. Cooke's section on "Mine Surveying" in *Practical Coal Mining* (edited by Prof. W. S. Boulton), vol. vi., p. 265, for further details of the method.

not point to that arrangement as the most equitable in the circumstances. Indicating average errors by suffixes as usual, the best shape for the triangle will be that in which $c_1 \div c$ is a minimum.

From equations (46) and (13) we have :—

$$\frac{c_1}{c} = \pm \frac{\sqrt{\{a_1^2(a-b\cos C)^2 + b_1^2(b-a\cos C)^2 + a^2b^2C_1^2\sin^2 C\}}}{a^2 + b^2 - 2ab\cos C} . \quad (48)$$

By differentiating with respect to C, and equating the result to zero, $\frac{c_1}{c}$ is found to be a minimum when $C = 180°$, and a maximum when $C = 0°$.

Similarly from equations (47) and (13) we get :—

$$\frac{c_1}{c} = \pm \frac{\sqrt{\{a_1^2\cos^2 B + b_1^2\cos^2 A + a^2B_1^2\sin^2 B + b^2A_1^2\sin^2 A\}}}{a\cos B + b\cos A} . \quad (49)$$

If the triangle is isosceles, $A = B$, $A_1 = B_1$, $a = b$, $a_1 = b_1$, and (49) reduces to—

$$\frac{c_1}{c} = \pm \sqrt{\left\{\frac{a_1^2}{2a^2} + \frac{A_1^2\tan^2 A}{2}\right\}},$$

which is a minimum when $A = 0°$, and a maximum when $A = 90°$.

Therefore, the best-shaped triangle, whether solved by eq. (46) or eq. (47), is the isosceles form in which the angle between the two measured parts of the base is nearly 180° : or, to put it in another way, a straight base is better than a broken one.

The average fractional error in the virtual base becomes very high when the angle opposite is small. Such a triangle, besides being of bad shape, is also uneconomical, since a comparatively large amount of linear measurement has to be made for a very short virtual base.

A more detailed analysis of eq. (48) shows that when C approaches 180° the effect of error in that angle is inappreciable as compared with that of the linear errors. When the triangle is isosceles, and $a_1 = b_1$, that equation becomes—

$$\frac{c_1}{c} = \pm \sqrt{\left\{\frac{1}{2}\left(\frac{a_1}{a}\right)^2 + \frac{C_1^2}{4}\left(\frac{1 + \cos C}{1 - \cos C}\right)\right\}}, \quad . \quad (50)$$

which shows the effect of linear error to be independent of the angle, C, and the effect of angular error independent of the lengths.

Taking the average fractional error $\left(\frac{a_1}{a}\right)$ in measuring the portion a as $\pm 1 \div 50{,}000$, and C_1 as ± 15 seconds ($10^{-5} \times 7{\cdot}27$ radians), and applying the criterion of negligibility of § 20, we find the effect of angular error to be inappreciable as compared with that of linear error when—

$$\frac{C_1^2}{4}\left(\frac{1 + \cos C}{1 - \cos C}\right) \leqq \frac{1}{9}\left\{\frac{1}{2} \cdot \left(\frac{a_1}{a}\right)^2\right\},$$

or when $C \geqq 165°$ about.

Providing, then, that the two parts of the broken base contain an angle between 165° and 180°, and that they are not far from equal in length, the surveyor need not put himself about in measuring that angle, for, in ordinary circumstances he may be sure, if the angle is measured with the same care as any other triangulation angle, that the error involved in it will be negligible in its influence on the calculated virtual base.

27. **The Case of a Triangle in which the Three Sides are measured for the Purpose of calculating One Angle.**—The formula for solving such a triangle is—

$$\tan \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}},$$

in which $s = \frac{1}{2}(a + b + c)$. As this formula is derived from—

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad . \quad . \quad (51)$$

the latter form will be used here as more suited to the purpose in view.

In determining the best shape for such a triangle it is necessary to consider each side subject to errors in measurement, and to find the effect of such errors, acting in combination, on the calculated value of the angle A.

From (51) and (13) we have, after simplification—

$$A_1 = \pm \sqrt{\left\{ \frac{4a^2b^2c^2a_1^2 + c^2b_1^2(b^2 - c^2 + a^2)^2 + b^2c_1^2(c^2 - b^2 + a^2)^2}{b^2c^2(2b^2c^2 + 2a^2b^2 + 2a^2c^2 - a^4 - b^4 - c^4)} \right\}} . \quad (52)$$

Inasmuch as (51) is symmetrical in b and c, we might infer that in the best-shaped triangle these two lines will be equal in length, and subsequent work will show that inference to be correct.

In the two instances given below, in which use is made in practice of this type of triangle, the lengths of the sides are short, none exceeding one tape-length. We may, therefore, say $a_1 = b_1 = c_1$ if the sides are measured with equal care (see § 28).

Hence, when $b = c$, eq. (52) reduces to—

$$A_1 = \pm \frac{a_1}{b} \sqrt{\left\{ \frac{4\,b^2 + 2\,a^2}{4\,b^2 - a^2} \right\}} . \quad (53)$$

Thus A_1 assumes a minimum value—namely, $\pm \frac{a_1}{b}$, when a is zero. That is to say, theoretically the best triangle is that in which the angle to be calculated is zero and the sides embracing it are equal in length.

We may now look to the practical aspect of the problem and see if this result gives any real help.

Mine surveying supplies two instances where the sides of a triangle are measured for the purpose of obtaining one of the angles. One of these is in a peculiar mode of shaft-connection, and the other in a method of underground traversing introduced by Mr. F. L. Burr, and first explained in a paper he read before the Lake Superior Mining Institute in October, 1905.

Let A and C (Fig. 11) represent two wires hung in a vertical shaft, and let it be assumed that the bearing of the line A C has already been ascertained by means of observations at the surface. It may have been found that, in order to get A C of sufficient length, the wires had to be hung in the end compartments of the shaft—a position which often precludes either the placing of a theodolite underground exactly in line with A C, or the use of the narrow triangle of Weisbach. We already know, by § 25, that if the theodolite were set in a position such as B and sights taken to the wires, grave error would probably enter into the calculated bearing of A B or C B, owing to the bad shape of the triangle. Sometimes, in circumstances of this kind, the idea of sighting the wires directly is abandoned; a pin is placed at B; the three sides, a, b, and c, are measured very carefully by the steel tape; the angle at A is calculated, and so the bearing of A B obtained. The theodolite is planted at D, and moved until exactly in the line A B produced. The line A D then forms the first draft of the underground traverse.

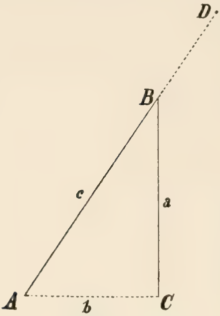

Fig. 11.

According to the result just derived (p. 57), the triangle A B C is best conditioned when B coincides with C—*i.e.*, when D is in A C produced. Now, if D could be brought into line, or nearly into line, with A and C, there would be no need to use this method at all. Therefore, with the case in point, the theoretical best shape has little or no practical value.

Burr's method of traversing is a rough process of surveying secondary mine openings, where there is magnetic attraction present, and where the theodolite is unnecessarily refined and quite possibly too bulky and delicate to take into the low and tortuous roads in question. If in nothing else, the method is of interest in that it requires the absolute minimum of equipment—namely, a roll of string, a few nails, and a tape.* It has evidently been suggested by the mode of using the German dial, or hanging compass, strings being stretched from side to side of the roadway in the manner shown in Fig. 12. The lines P A, A Q, Q R, etc., of the traverse are measured, while, to obtain an angle such as P A Q, riders of copper wire or other suitable marks are placed on the strings at B and C at distances of 4 or 5 feet from A. The three sides of the triangle A B C are then taped, and the angle at A calculated. The other angles of the traverse are obtained in like fashion.

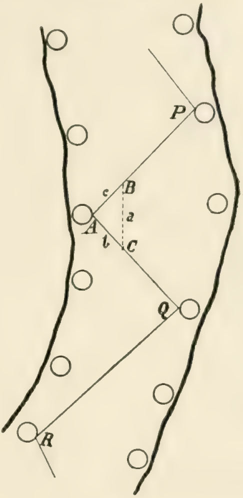

Fig. 12.

The absurdity of using the best-shaped triangle is still more evident here, for if the angles A, Q, R, etc., were made very small the number of lines required would become very great. The most economical shape for such a triangle as A B C would be that in which A

* To these a clinometer must be added if the road is inclined

is 180°, or $a = 2b$; yet eq. (53) shows the average error A_1 to be excessive under that condition. Burr's method, therefore, suffers from the disadvantage that the best shape for the triangles used in it is the least economical, and the most economical the worst shape.

The fact that the best form cannot be applied in either of these cases leads us to look for shapes, which, if not so perfect from the theoretical point of view, are such as will prevent the average error in the calculated angle from being unduly large, while being suitable for application in practice.*

Nothing more need be said about Burr's method than that the requirements as to economy will have to be balanced against those of accuracy. The most suitable shape of a triangle in any case arising in practice will be a compromise, and, by making the angle as nearly as possible a right angle, about equal weight is given to each of these antagonising factors.† The method can, of course, never be more than a rough one.

The shaft-connection problem is, however, of sufficient interest to warrant further analysis. The best shape, when b is made equal to c, having turned out unsatisfactory, let us find the most suitable shape when $c = 2b$.

* Before dismissing the theoretical best shape, it may be mentioned that if in Weisbach's triangle (Fig. 8) a point were ranged by the theodolite exactly in the line C B and alongside A, it would define, with the wires, a triangle almost conforming with this theoretical best shape. By measuring the sides of that triangle carefully, a perfectly independent result for the angle B could be obtained by calculation. This result, though it could not usually be taken as of equal weight with that derived from the ordinary solution of a Weisbach triangle, would be valuable as a check.

† The fact that the triangle is only one of a number, and that the error resulting in an angle has a cumulative effect, is not dealt with here. After having read the next two chapters, the reader may take as an exercise to determine the shape for the triangles in Burr's method, such that, in a traverse along a straight road of given length and width, the average total error at the end of the traverse will be a minimum.

When $c = 2b$, eq. (52) becomes—

$$A_1 = \pm \frac{a_1}{b} \sqrt{\left\{ \frac{5a^4 - 2a^2b^2 + 45b^4}{4(10a^2b^2 - a^4 - 9b^4)} \right\}}.$$

Let $a = pb$; then this equation can be written—

$$A_1 = \pm \frac{a_1}{b} \sqrt{\left\{ \frac{5p^4 - 2p^2 + 45}{4(10p^2 - p^4 - 9)} \right\}}. \quad . \quad . \quad (54)$$

This will be a minimum when the part under the root-sign is a minimum. By taking that part, differentiating it with respect to p, and equating the result to zero, it is found that $p = \sqrt{3}$ makes it a minimum. That is to say, when $c = 2b$, the best shape is given by $a = b\sqrt{3}$. Such a triangle has a right angle at C, and an angle of 60° at A.

Now, consider the case when $c = b\sqrt{2}$. Letting $a = pb$, and proceeding as before, we get—

$$A_1 = \pm \frac{a_1}{b} \sqrt{\left\{ \frac{3}{2}\left(\frac{p^4 + 2p^2 + 1}{6p^2 - p^4 - 1}\right) \right\}}, \quad . \quad (55)$$

which will be found to reach a minimum value when $p = 1$. The most satisfactory triangle is, therefore, that in which $a = b$, $c = b\sqrt{2}$, or one having A = B = 45°, and C = 90°.

Similarly, the case when $c = 3b$ gives—

$$A_1 = \pm \frac{a_1}{b} \sqrt{\left\{ \frac{10p^4 - 92p^2 + 640}{9(20p^2 - p^4 - 64)} \right\}}, \quad . \quad (56)$$

which is a minimum when $p = \sqrt{8}$—*i.e.*, when the triangle has a right angle at C.

These cases go towards proving the following general rule :—*When c is greater than b*, the shape giving rise to least error in the angle A is that in which C is 90°.

When $c = b/2$, we get—

$$A_1 = \pm \frac{a_1}{b} \sqrt{\left\{ \frac{80p^4 - 6p^2 + 45}{40p^2 - 16p^4 - 9} \right\}}, \quad . \quad (57)$$

which is smallest when $p = \frac{\sqrt{3}}{2}$; *i.e.*, when B = 90° and A = 60°. The following rule can be proved by taking a number of similar instances :—*When c is less than b, the shape giving rise to least error in A is that in which B is 90°.*

By substituting into the equations (53) to (57) inclusive, those respective values of $p\left(=\frac{b}{a}\right)$ which make A_1 a minimum, we obtain the results given in the table below :—

Relation between b and c.	Angles in the Best-shaped Triangle.			Minimum Value of Average Error (A_1) in Calculated Angle A.	Minimum Value of A_1 when $a_1 = \frac{1}{100}$ ft. and $b = 10$ ft.	Equation from which Minimum Value of A_1 derived.
	A	B	C			
(1)	(2)	(3)	(4)	(5)	(6)	(7)
				Radians.	Seconds.	
$c = b$, .	0°	90°	90°	$\pm a_1/b$	± 52	(53)
$c = b\sqrt{2}$, .	45°	45°	90°	$\pm 1{\cdot}22\ a_1/b$	± 63	(55)
$c = 2b$, .	60°	30°	90°	$\pm 1{\cdot}32\ a_1/b$	± 68	(54)
$c = 3b$, .	70° 32′	19° 28′	90°	$\pm 1{\cdot}37\ a_1/b$	± 71	(56)
$c = b/2$, .	60°	90°	30°	$\pm 2{\cdot}67\ a_1/b$	± 96	(57)

Column (5) shows that, while no arrangement gives such a low minimum value of A_1 as the theoretically best triangle, yet A_1 increases only slowly as c increases, providing C is kept about 90°. When c is made smaller than b, however, the value of A_1 soon assumes large dimensions, even when the favourable triangle having B = 90° is used. There is also another reason why c ought not to be made less than b, the distance between the wires—namely, that the operation of lining in the theodolite at D (Fig. 11) becomes difficult and inexact when the distance between the pin, B, and the wire, A, is short. Special care, then, must be taken to make this distance greater than that between the wires.

It is probable that the best results in practice will be attained when $C = 90°$, and c is between twice and thrice as long as b; for when such an arrangement is compared with one in which the pin is set nearer the wires (say with $c = b\sqrt{2}$), the small increase in A_1 is likely to be compensated by the greater precision of lining-in the instrument at D.

As it seems unlikely under the conditions obtaining in practice that the average error, a_1, in measuring one of the sides can be reduced much, if any, below $\pm \frac{1}{400}$ foot with a steel tape graduated to hundredths of a foot, even when sag is prevented, slope corrected for, and several readings taken in different parts of the tape, the accuracy attainable by this method of transferring bearing underground can never be great (see column (6)); therefore, it cannot be recommended.

CHAPTER IV.

THE PROPAGATION OF ERROR IN TRAVERSING.

"A knowledge of the probable error of making the various observations which together give the traverse-survey, and the way in which these affect the final result, is of use, not only in indicating the precision required in the instruments and operations for any particular survey, and, therefore, determining the instruments that should be used for it, and the time that should be spent; but it is of even more use as a test to apply to the results, to enable one to judge whether they are satisfactory, or whether any unusual error has been made. An examination of the various parts which together constitute the whole error in a single measurement, and the way in which these are carried through the survey, will often decide the best method of using a particular instrument."—*Prof. G. R. Thompson.*

CHAPTER IV.

THE PROPAGATION OF ERROR IN TRAVERSING.

28. **Accuracy of Linear Measurements.**—Errors in traversing fall into two classes :—

(*a*) Errors in measuring lengths ; and
(*b*) Errors in measuring angles.

This section is for the purpose of discussing the first of these classes.

At first sight it seems remarkable that, having regard to the simplicity of the measurement itself, and to the fact that the question has been closely considered by other writers, more difficulty should be encountered in assessing the average error of linear measurement than that due to other cause.

This difficulty is chiefly owing to the interference of cumulative error, which is far more troublesome to reduce to negligible amount than when angles are being determined. When one considers that the precautions, sometimes very elaborate, taken in measuring triangulation base-lines are largely directed against cumulative error, the reality of the difficulty when dealing with distances in traversing will be readily granted. These cumulative errors, as well as the accidental errors, are, moreover, very susceptible to the nature of the ground over which measurement is made. It matters little or nothing to the angles that the ground is rough and stony, but it is common knowledge that the accuracy of the linear measurements suffers by those circumstances. Another point militating against the exact determination of linear errors is that, while angles are measured by one person usually skilled in the work, to measure lengths by tape or chain requires two persons. Sometimes both these persons are skilled, sometimes one is skilled and the other not, while in many cases neither of them is sufficiently conversant with the work nor sufficiently realises its importance.

Since both ends of a line have the same "weight," it is of little use to read a steel tape with exactitude at one end if the man at the other is not taking proper care. The last trouble reaches a climax in British mine-surveying practice, where the surveyor is often obliged to leave the chaining to a couple of workmen.

The value of introducing more refined methods of linear measurement is, indeed, doubtful unless the machinery of making the measurements is correspondingly improved. The subtense method of measuring traverse lines, especially in mine surveying where the graduated rod may often be hung from a staple in the roof—when it becomes self-plumbing—has much to commend it on this account. Although not more accurate than good chaining, it places the linear measurements in skilled hands, and for that reason is likely to give more concordant results than the ordinary methods into which the factor of ignorance enters so largely. Moreover, the accuracy of tacheometric observations is influenced only to a slight degree by the nature of the ground.

Again, while experience in triangulation, if correctly analysed, gives information of the first importance as to the errors to be expected in angular measurement in traversing, there is no operation in ordinary practice giving evidence of like value in regard to distances.

In face of all these difficulties, it cannot be helped that the rules for finding average errors in linear measure are only rough. We must rest satisfied if they are near enough for practical purposes.

It is necessary to distinguish between—

(1) Lengths so short that one application of the steel-band or chain is sufficient to cover them ; and

(2) Lengths so long that several applications of the band, chain, or measuring-rod are required.

The second type is the commoner in traversing, but the first occurs sufficiently often, especially in mine surveying, to merit attention.

(1) In the case of a line less than one chain long, measured by a chain *to the nearest link*, assuming that sagging is prevented and slope corrected for, the accidental errors that chiefly matter are those of reading and graduation of the chain. The error at the zero end, due to the handle not being held exactly at the point, unless the person there is excessively careless, is so much less than that due to reading as to be negligible under the criterion of § 20. The error of graduation may be appreciable if the chain has frequently been corrected in the usual way by taking out some of the small oval rings, and if the rings were not taken from points roughly equidistant along the chain.* The error due to that cause, however, cannot in general be taken as being in any way proportional to the length measured, for the 60-mark has an equal chance with, say, the 40-mark of being a little out of place.

The reading error, also, is independent of the length of such a line. The chain being read to the nearest link, a distance of 70·2 or 70·4 would be recorded as 70, while one of 70·6 or 70·8 would be entered as 71. We therefore see that the reading error ranges from half a link to zero. Now, there is an equal likelihood of any size of error between those limits ; therefore, the average error of reading is $\pm$ 0·25 link. Making allowance for the other sources of accidental error, the average error in measuring a line less than one chain length by means of a chain, reading to single links, will usually be between $\pm$ 0·25

* It must be remembered that we are dealing here only with accidental error. The chain is assumed of the correct total length.

and $\pm$ 0·3 link, and it is independent of the length of the line.

In the case of a steel band graduated to hundredths of a foot, and read to the nearest division, the average error in reading one end will be, by similar reasoning, $\pm \frac{1}{400}$ foot. The error at the zero end cannot now be assumed negligible, and must be taken as of the same magnitude. Considering both ends, the total reading error will be $\pm \sqrt{(\frac{1}{400})^2 + (\frac{1}{400})^2}$, or $\pm$ 0·0035 foot (§ 11). The error due to imperfect graduation will usually be too small to be of account. Inasmuch as the tension and temperature of the tape are not measured in traversing, it will be as well, in order to allow for error arising out of that fact, to state the average error in a single measurement of such a line as about $\pm$ 0·004 foot. It is so nearly independent of the length as to permit us to take it so in practice.

The above represents the most favourable case, in which the tape is laid on a smooth and even surface, and slope, if any, carefully taken into account. If the band is allowed to sag, or if it lies on an undulating surface, cumulative error at once makes itself felt.

(2) The case of a line considerably longer than the chain, tape, or rod now falls to be considered. Such a line requires several applications of the instrument to cover it, and a new source of error enters—namely, that of marking the end-points of each chain, tape, or rod-length. Marking involves both the act of reading and the act of fixing the mark ; therefore, the average error of marking must be greater, and is generally considerably greater, than that of reading. In a favourable case we shall probably not be far from the truth if we take it as thrice as much —*i.e.*, $\pm \frac{3}{400}$ foot for a steel tape in ordinary traversing.

In making one application of a steel band resting on a uniform level surface, an error will arise due to marking at one end and to reading the zero-division at the other.

and these in combination will have an average magnitude of about $\pm \sqrt{(\frac{3}{400})^2 + (\frac{1}{400})^2} = \pm 0{\cdot}008$ foot. If a line involves n applications of the band, each affected by this average error, the average error, l, in the whole measurement will be $\pm 0{\cdot}008 \sqrt{n}$. It is seen that the error can be materially reduced by using a longer tape, which requires fewer applications to cover the distance in question. This fact is widely admitted in practice, and gives the reason why long tapes are preferred in making specially delicate linear measurements, as, for example, those of triangulation base-lines.

In the case of a 100-foot tape—a length common in British and Colonial practice—a distance of L feet would require L ÷ 100 applications, and for that special instrument the above result may be written :—

$$l = \pm 0{\cdot}0008 \sqrt{L}, \quad . \quad . \quad . \quad (58)$$

This gives $l = \pm 0{\cdot}016$ foot for a distance of 400 feet, or an average fractional error of 1 in 25,000 in that distance.

Now, we know from experience that we cannot expect such a high degree of accuracy as this with a 100-foot tape. We have indeed been discussing the most favourable case possible—where the ground is smooth and level, and both of the persons making the measurement skilled surveyors. The difficulties spoken of at the beginning of this section now begin to exert their influence, and to carry the matter further by *a priori* reasoning is impossible. Recourse must, therefore, be had to experiment to determine how eq. (58) ought to be modified to suit practical conditions.

Perhaps the most valuable experiments made in this connection were those of Professor Lorber, of the Leoben School of Mines. They were quoted by Brough.* From over 6,000 observations Prof. Lorber deduced the mean-

* B. H. Brough, *A Treatise on Mine Surveying*, Ninth Edition, p. 24.

square error affecting linear measurements of different kinds. The following table summarises his results, the mean-square being converted into average error by the aid of eq. (8) :—

Two rods along a stretched cord, .	$l = \pm 0{\cdot}000427\sqrt{L}$ *
Two rods used without a cord, .	$l = \pm 0{\cdot}000740\sqrt{L}$
Field compasses, . . .	$l = \pm 0{\cdot}00169\sqrt{L}$
Steel band, ...	$l = \pm 0{\cdot}00172\sqrt{L}$
Chain, ...	$l = \pm 0{\cdot}00239\sqrt{L}$
Measuring wheel, . . .	$l = \pm 0{\cdot}00287\sqrt{L}$

Of these, the results most of interest to English readers are those dealing with the steel band and the chain. It is unfortunate that the lengths of the band and chain used are not quoted, for we have already seen the error to depend, not only on the class of instrument employed, but also on its length. A 100-foot tape, for instance, cannot be expected to give such accurate results as a 300-foot tape. Setting aside this objection, the above results show the average error in chaining to be approximately 1·5 times that in using the steel band. Hence, from § 9, we obtain the following rough rule :—

$$\left.\begin{matrix}\textit{Weight of a steel-tape}\\ \textit{measurement}\end{matrix}\right\} : \left\{\begin{matrix}\textit{weight of a chain}\\ \textit{measurement}\end{matrix}\right\} :: 2\tfrac{1}{4} : 1.$$

Lorber also made an effort to determine the effects of cumulative errors (l_c) in his measurements—that is, such errors as those due to sagging, unstandard temperature, and imperfect alignment. These errors, which are directly proportional to the length measured, were found to be as follows :—

* W. Jordan gives a rule for this case, for lengths between 9 and 28 metres, which is equivalent to : $l = \pm {\cdot}00029\sqrt{L}$.

Two rods along a stretched cord, .	l_c *very small.*
Two rods used without a cord, .	$l_c = - 0{\cdot}00008$ L.
Steel band, ...	$l_c = - 0{\cdot}00032$ L.
Chain, ...	$l_c = + 0{\cdot}00046$ L.
Field compasses, . . .	$l_c = - 0{\cdot}00079$ L.

Although the effect of the latter class of errors is to give results too high for the chain and too low for the band, yet it will be noticed that their respective magnitudes are again nearly in the ratio of 3 to 2.

Cumulative error in linear measurements can only be brought down to negligible proportions with great difficulty; therefore, in ordinary work, such as traversing, regard must be paid to it. The usual way of taking it into account is to assume all error to obey the rule—

$$l = \pm \mathrm{K} \sqrt{\mathrm{L}}, \qquad . \quad . \quad . \quad (59)$$

and to give to the coefficient, K, values depending on the nature of the work performed, the kind of instrument employed, the skill of the operators, and the average length of the lines measured: making K always of sufficient size to cover the cumulative error to be expected. It is evident from the character of Lorber's results that the cumulative error cannot be accurately dealt with by such a method, and, indeed, it must be admitted that the convention is open to attack from the mathematical standpoint. There seems little doubt, nevertheless, that the method is the best the circumstances permit of; for if the linear errors affecting, say, measurements by the steel tape were stated in the form—

$$l = \pm \mathrm{K}_a \sqrt{\mathrm{L}} - \mathrm{K}_b \, \mathrm{L},$$

it would be hardly possible to handle them mathematically, since the theory of errors cannot be applied to a composite error, of which one part is average error with the plus or minus sign, and the second part actual error without the double sign. On the other hand, if we refuse to take the effects of cumulative error into consideration, results are obtained which prove unlike those obtained from experience—and that, after all, must be the final test.

Before going on to state one or two expressions designed to cover, as well as possible, all the error to be expected in measuring a line, it seems not inadvisable to make mention of a criterion of accuracy common among surveyors. A surveyor speaks, say, of getting "an accuracy of 1 in 1,000" in measuring over certain ground with a chain. Excusing the confounding of the terms "accuracy" and "error," the expression is faulty—first, in that it usually conveys the impression that lines of a variety of lengths have been measured with the fractional error stated, and secondly, since it is often taken to mean that if one of the lines were to be measured again its error would, in all likelihood, be *less than* one-thousandth of its length. As a matter of fact—to deal with the last objection first—to secure odds of, say, 10 to 1 in favour of the actual error of a measurement falling within one-thousandth of its length would require the average error of a single measurement to be about $\pm 1 \div 2{,}100$ (see Table III., Appendix),—which is a much higher precision than is really meant. If the statement in question is based on experience, it will probably have been derived from finding that *on the average* the error was about one-thousandth of the length of a line, and actually the ratio is one of average fractional error, and should, therefore, not be taken in the sense of a maximum. The nature of the other objection is evident when one considers the form of eq. (59), in which the error is taken to be proportional to the root of the length rather than to the length itself. The statement would be less faulty if it were meant to apply to lengths between certain limits. As eq. (59)

has been shown to be somewhat conventional, there would be no necessity for those limits to be inconveniently close.

Notwithstanding the objections named, the fractional error has so many advantages as a criterion of accuracy in surveying, and is so universally used, that the author would be the last to recommend its abolition.

The following special forms of the general expression eq. (59) were given by Lorber* for "unfavourable cases":—

Two wooden 4-metre rods along a stretched cord, . . .	$l = \pm\, 0{\cdot}000427\sqrt{L}$.
Two wooden 4-metre rods used without a cord, . . .	$l = \pm\, 0{\cdot}00266\sqrt{L}$.
Steel band, ...	$l = \pm\, 0{\cdot}00621\sqrt{L}$.

For these three methods of measurement, the average errors may be taken as lying between the values just stated and those quoted by Brough and given in the table, p. 72, but in practice they will usually incline more towards the former than the latter. For land surveying practice in traversing it will probably not be far from the mark to assume the average error in using the steel band to lie between $l = \pm\, 0{\cdot}004\sqrt{L}$ and $l = \pm\, 0{\cdot}0062\sqrt{L}$, remembering that in specially favourable instances it may be less than the former, and over bad ground greater than the latter. The first of these gives an average fractional error of 1 in 5,000, and the second one of 1 in 3,200 for lines 400 feet in length.

In the mine the conditions are usually worse than those obtaining at the surface, and l for the 100 ft. steel tape is taken as $\pm\, 0{\cdot}0063\sqrt{L}$ in several mine-surveying examples worked out in this and the next chapter.

It is hoped that the reader will not apply these rules

* From Höfer's "Taschenbuch für Bergmänner," 1911, p. 1007.

to cases arising in his own practice without first putting them, if possible, to the test of his own experience.

29. **The Average Error due to Imperfect Centring.**—In discussing the errors affecting angular measurements in traversing, regard must be had to four classes of accidental angular error, namely :—

(*a*) Error due to centring the instrument imperfectly ;

(*b*) Error in sighting ;

(*c*) Error in reading the verniers or needle ;

(*d*) Levelling error, and other error due to instrumental imperfections.

It is the purpose of this section to consider the first of these.

In practice we speak of centring to within such-and-such a distance, meaning that we permit centring displacements of less than that distance, but disallow any of greater magnitude. Thus in the first instance we actually know the *maximum permissible displacement.* Our immediate object is to determine the angular error resulting on the average from displacements in centre, which may be anything between that maximum amount and zero. Unless the average angular error due to imperfect centring is ascertained. it is impossible justly to compare it with those other types enumerated above.

Let γ be the maximum permitted eccentricity. Let O (Fig. 13) represent a traverse station, and P_1 and P_2 the adjacent stations, the traverse angle or polygonal angle being T. E a F is a circle about O as centre, and of radius γ. The actual point over which the instrument is set might lie anywhere within this circle, and—remembering how centring is performed—the probability of it being over any one point (such as N) within the circle is equal to that of it being over any other.

When the instrument stands over the point N, the angle $P_1 N P_2$ (or S) will be measured instead of the

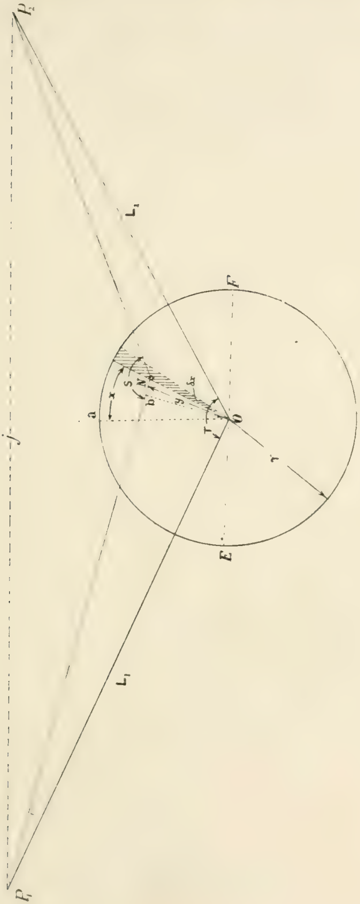

Fig. 13.

correct angle, T. The angular error due to this particular setting is, therefore, (S − T).

Bisect the angle T by the line O a. Referring the position of N to this bisectrix, let the angle a O N be termed x, and the distance O N, y.

Then, O b being drawn at right angles to P_1 N, we have—

$$\text{Angle } O P_1 N = \frac{O b}{O P_1} = \frac{O b}{L_1} = \frac{y}{L_1} \sin\left(\frac{T}{2} + x + O P_1 N\right);$$

or since O P_1N is very small—

$$O P_1 N = \frac{y}{L_1} \sin\left(\frac{T}{2} + x\right).$$

$$\text{Similarly,} \qquad O P_2 N = \frac{y}{L_2} \sin\left(\frac{T}{2} - x\right).$$

But an inspection of the figure shows that—

$$S - T = \pm (O P_1 N + O P_2 N).$$

Therefore,

$$S - T = \pm \left\{ \frac{y}{L_1} \sin\left(\frac{T}{2} + x\right) + \frac{y}{L_2} \sin\left(\frac{T}{2} - x\right) \right\}. \quad (60)$$

This equation is quite general for all positions of N within the circle.

There must be a line, E O F, at all points along which (S − T) is zero. As L_1 and L_2 are very large compared with the size of the circle, E O F may be taken as a straight line. Equating (60) to zero, we obtain for any point on E O F—

$$\frac{y}{L_1} \sin\left(\frac{T}{2} + x_0\right) = -\frac{y}{L_2} \sin\left(\frac{T}{2} - x_0\right),$$

$$\text{or} \qquad x_0 = \tan^{-1}\left\{ \left(\frac{L_1 + L_2}{L_1 - L_2}\right) \tan\frac{T}{2} \right\}, \quad (61)$$

where x_0 is the angle between the lines O a and E F. Hence, when x is less than x_0 or greater than $(180° + x_0)$, the error (S − T) will have a positive sign, while for all values of x between x_0 and $(180° + x_0)$ it will have a negative one.

Consider an elemental segment (shown shaded), containing N, and lying within the semicircle E a F, in which (S − T) is positive. Let the segmental angle be δx.

Let all such points as N stand in a very small rectangular area $\delta y \times \delta z$, of which δz is the circumferential dimension.

The number of points such as N *in the semicircle*

$$\text{E}\,a\,\text{F} = \frac{\pi \gamma^2}{2} \times \frac{1}{\delta y \,.\, \delta z}. \qquad (62)$$

Also, there are $\frac{y \,.\, \delta x}{\delta z}$ points such as N in the elemental segment at a distance y from the centre. Therefore, making use of (60) and (61), the sum of the angular errors for all the points in the semicircle E a F is given by—

$$\Sigma\left[\Sigma\left[y\left\{\frac{\sin\left(\frac{T}{2}+x\right)}{L_1}+\frac{\sin\left(\frac{T}{2}-x\right)}{L_2}\right\}y\,\frac{\delta x}{\delta z}\right]_{y=0}^{y=\gamma}\right]_{x=\pi+\tan^{-1}\left\{\frac{L_1+L_2}{L_1-L_2}\,.\,\tan\frac{T}{2}\right\}}^{x=\tan^{-1}\left\{\frac{L_1+L_2}{L_1-L_2}\,.\,\tan\frac{T}{2}\right\}}$$

By dividing this total by the number of points, as given by (62), we obtain the average angular error (see eq. (3)). Proceeding to the limit, this may be written as—

Average angular centring error over the semicircle E a F

$$= \frac{2}{\pi\gamma^2}\int_{\pi\tan^{-1}\left\{\frac{L_1+L_2}{L_1-L_2}\,.\,\tan\frac{T}{2}\right\}}^{\tan^{-1}\left\{\frac{L_1+L_2}{L_1-L_2}\,.\,\tan\frac{T}{2}\right\}}\int_0^{\gamma}\left[y^2\left\{\frac{\sin\left(\frac{T}{2}+x\right)}{L_1}+\frac{\sin\left(\frac{T}{2}-x\right)}{L_2}\right\}dx\,.\,dy\right]$$

$$= \frac{4\gamma}{3\pi}\sqrt{\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}-\frac{2\cos T}{L_1 L_2}\right)}.$$

The same integration-process determines the average error over the other half-circle, the limits being reversed.

Hence the result is the same, but with the negative sign. Therefore, we can say in general—

$$\left.\begin{array}{l}\textit{The average angular}\\ \quad\textit{error due to im-}\\ \quad\textit{perfect centring}\end{array}\right\} = \pm\frac{4\gamma}{3\pi}\sqrt{\left(\frac{1}{L_1^{\,2}}+\frac{1}{L_2^{\,2}}-\frac{2\cos T}{L_1\,L_2}\right)}. \quad (63)$$

A similar, but simpler, train of reasoning,* based on the assumption that the *average* centring displacement, r, is known instead of the maximum permissible displacement, γ, gives—

$$\left.\begin{array}{l}\textit{The average angular}\\ \quad\textit{error due to im-}\\ \quad\textit{perfect centring}\end{array}\right\} = \pm\frac{2\,r}{\pi}\sqrt{\left(\frac{1}{L_1^{\,2}}+\frac{1}{L_2^{\,2}}-\frac{2\cos T}{L_1\,L_2}\right)}. \quad (64)$$

Therefore, it is evident that—

$$r = \frac{2\,\gamma}{3}\dagger. \quad . \quad . \quad . \quad (65)$$

For example, if displacements of centre not greater than $\frac{3}{4}$ inch are permitted, the actual eccentricity in any case may be anything between $\frac{3}{4}$ inch and zero, and the average eccentricity will be $\frac{1}{2}$ inch.

Equation (63) shows the angular error due to imperfect centring to depend on the size of the traverse angle, T, the error being a maximum when T is 180° and a minimum when T is zero.

It also gives expression to a fact long known in practice —namely, that a centring displacement has more effect on a short than on a long line, and further states the angular error to be directly proportional to the magnitude of the displacement.

30. **Transectors and Reciprocal Transectors.**—The tables of squares and reciprocals of squares given in the Appendix considerably facilitate the calculation of the angular error due to imperfect centring from eqs. (63) or (64);

* *Trans. Roy. Soc. Edin.*, vol. xlvii., Part IV., p. 851.

† See footnote, p. 6.

but nevertheless it remains the most tedious of all the calculations to be made in studying the propagation of error in actual traverses. By employing either of the methods described in this section, however, the time required to evaluate the centring error is reduced, and simple equations, admitting directly of the use of logarithms, are formed to take the place of (63) and (64).

Equation (63) may be written in the form—

$$\left.\begin{array}{l}\text{Average error due to}\\ \text{imperfect centring}\end{array}\right\} = \pm \frac{4\gamma}{3\pi L_1 L_2} \sqrt{L_1^2 + L_2^2 - 2 L_1 L_2 \cos T},$$

and this may be re-stated as—

$$= \pm \frac{4\gamma}{3\pi L_1 L_2} j, \quad . \quad . \quad . \quad (66)$$

in which j is the distance between the two stations adjacent to the one under consideration. Such a distance is termed by the author a *transector*. For example, $P_1 P_2$ (Fig. 13) is the transector of station O.

To determine transectors necessitates that the traverse be plotted roughly by means of a protractor, and the required spans measured off to scale, or that some other plan be available (quite a rough one will serve) on which the station-points can be marked. Fig. 17 shows an actual closed traverse across which the transectors have been drawn.

Still another manner of writing (63) is—

$$\textit{Average error due to imperfect centring} = \pm \frac{4\gamma}{3\pi}\omega, \quad (67)$$

which is the simplest form of all. The author terms ω of (67) the *reciprocal transector;* it is the distance from a station to the next but one when the *reciprocals* of the traverse lines are plotted instead of the lines themselves. The reciprocals are quickly read off from Table I. of the Appendix.

Fig. 19 illustrates the use of reciprocal transectors. The angles in Figs. 18 and 19 are the same, but while the lines of the traverse are shown in the former figure to scale, the reciprocals of their lengths are plotted in the latter.

Although the use of reciprocal transectors makes the calculation of the average centring error one of great simplicity, yet the method involves the plotting of the reciprocal traverse, which is of no further service once the reciprocal transectors have been measured from it. A plot of the normal traverse (such as Fig. 17), on the other hand, serves as a useful guide or index during the calculation of co-ordinates, and possibly during the field work, and so has a wider value. For this reason, no doubt, many, like the author, will prefer to use transectors rather than reciprocal transectors in determining average centring error.

31. **Errors of Sighting, Reading, etc.**—Assuming that a theodolite in good adjustment is employed, and that a method of measuring angles is used by which the effects of the main instrumental imperfections are reduced to small proportions, an angular measurement will be affected with errors of sighting and reading the instrument, and also with other minor, and more or less obscure, instrumental errors—quite apart from the error due to faulty centring discussed above. With the exception of that of sighting, these errors are manifestly independent of the lengths of the lines, and can, therefore, be taken as having a constant average value for all lines.

The error of sighting is more difficult of treatment. It is very certain that the average sighting error over an exceptionally short line, such as one 15 feet in length, is greater than for one of average length, of say 200 feet; yet, on the other hand, there would not seem to be any sensible difference in accuracy of sighting as between lines, say, 200 and 400 feet long. Moreover, atmospheric unsteadiness and haze, which interfere to a greater extent with long lines, tend to nullify the increase in precision resulting from the finer definition of an object sighted at the extremity of a long draft, and will indeed outweigh it under certain atmospheric conditions. It would, therefore, be incorrect to assume that sighting errors continue to shrink as the lines increase in length above a certain point, and we shall approximate closely to the truth if we take this class of error as having a constant average value for all lines except the very shortest.

Let the average value of the combined errors of sighting, reading, etc., be $\pm v$ radians.

32. **Average Error in Traverse Angles.**—Leaving exceptionally short lines out of consideration, the error, t, in a traverse angle, T, is thus compounded of a constant component, v, and a variable component—namely, the error due to imperfect centring.

Combining these, we have, by eq. (15)—

$$t = \pm\sqrt{\left\{v^2 + \frac{16\,\gamma^2}{9\,\pi^2}\left(\frac{1}{L_1^2} + \frac{1}{L_2^2} - \frac{2\cos T}{L_1 L_2}\right)\right\}}. \quad (68)$$

The values of v and γ vary greatly with different observers and instruments. To a surveyor it is not of the first importance to obtain results in which *general* average values of these quantities are assumed; in order to make fullest use of eq. (68)—or, indeed, of any relation in the book—it is necessary that he should determine, as nearly as possible, his own average errors as given by his own instruments.

Experience in triangulation is the best guide as to the magnitude of the sighting-and-reading error, v, for a triangulation line (as will shortly be shown) is almost always so long that the angular effect of a small centring displacement is negligible; hence, in measuring a triangulation angle, the second term on the right-hand side of (68) disappears, leaving $\pm v$ as the error in angle.

The method of analysing triangulation experience in order to ascertain the error in a single measurement of an angle is given in § 11 (*c*). By that method it may be shown that, if three reiterations on each face were to result in an average error of summation of the three angles of a triangle of $\pm$ 12 seconds, the average error in determining *one* angle from the mean of single face-left and face-right measurements by the same theodolite would also be $\pm$ 12 seconds, and this would be taken as v for the particular instrumental method named.

By examining one's own practice, γ, the maximum centring displacement, is easily ascertained; but one should guard against taking it too low. The permitted displacement depends on the class of work undertaken and the character of the instrument. Owing to the greater length of time needed to centre closely, a larger eccentricity is generally allowed when the theodolite is unprovided with a sliding stage than when it has that useful addition. Centring a theodolite over a point on the ground in the ordinary way is not usually so precise as centring it under a station in the roof of a mine (see frontispiece), where the point of the plummet hangs above the true centre of the instrument, and where it is closely in view while moving the sliding head, without having to bend. In this connection it may be mentioned that the special form of "mechanical stage" shown in Fig. 14,* by which the instrument may be moved in two directions at right angles after the manner of the tool-rest of a lathe, not only gives more accurate centring, but also possesses the advantage that the adjusting screws may be turned and the centring done while bending down to see the

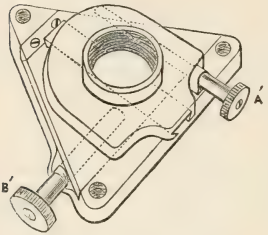

Fig. 14.

* *Vide* W. F. Stanley's "Surveying and Levelling Instruments," 1901, p. 248.

plumb-bob clearly. With ordinary forms of moving heads the operator must stand up when shifting the stage.*

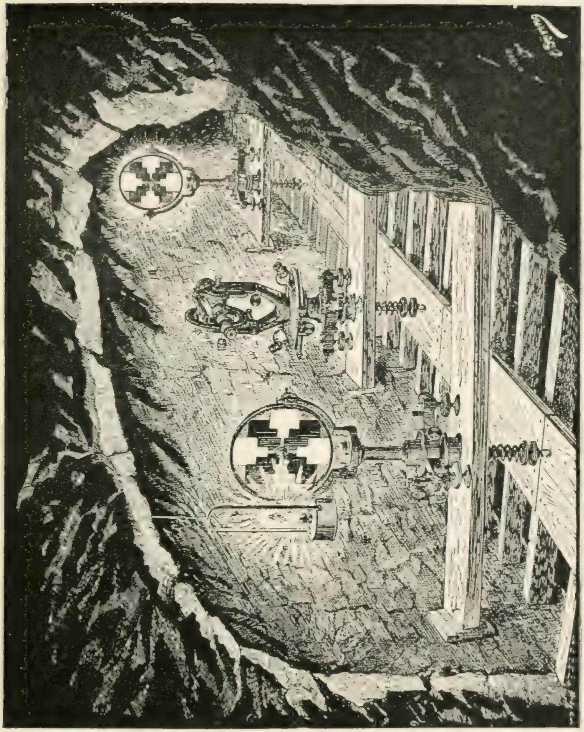

Fig. 15.

In any three-stand method centring error is eliminated. Fig. 15 shows the Breithaupt method of underground

* A recent form of bob, the invention of Mr. W. H. Shortt allows of underneath plumbing without stooping. The station-point is visible through the hollow centre of the bob.

traversing. In connection with the special form of theodolite there are three levelling heads, and, at any time during the traverse, the instrument rests on one of these, the second defines the back station, and the third the fore station. By setting targets at the fore and back stations in the manner shown, sights are able to be taken directly to the point at which the instrument last stood, and also to the point where it will next stand. When shifting the theodolite it is lifted out of the levelling head, carried forward, and set in the levelling head at the fore station, where it takes the place of the target. The operation of centring is here abolished — a marked advantage, especially on the short drafts which so often occur in the mine.

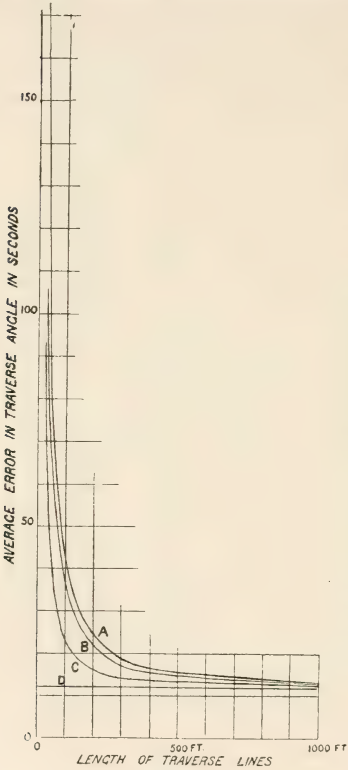

Fig. 16.

Once the maximum centring displacement, γ, and sighting-and-reading error, v, have been determined, the surveyor can make use of eq. (68) to ascertain whether he is justified in neglecting errors of these magnitudes in the case of any traverse angle he may select.

The relative effects of v and γ are perhaps most clearly elucidated by plotting curves for specific values of those quantities. The curves of Fig. 16 are constructed from relation (68), 0·024 foot having been selected as γ * and $\pm$ 12 seconds as v. The latter value probably approximates closely to a general average for traversing when a reliable 5-inch theodolite is used and when the angle is obtained by taking the mean of single face-left and face-right measurements, while the former is, if anything, slightly more than that permitted in ordinary practice.

These graphs, besides illustrating the decrease in importance of centring displacements as the traverse angle becomes smaller, also show that, while the centring error has a preponderant influence when the lines are short, its effect rapidly diminishes as they become longer.

The necessity of close centring on short lines is generally admitted, and in emphasising this fact, especially for traverse angles approaching 180°, lies the chief practical value of these curves. They illustrate, for example, that with a traverse angle of about 180°, and lines of less than 100 feet, nothing will be gained in accuracy by an endeavour to reduce the average sighting-and-reading error, so long as the average centring displacement remains $\frac{3}{16}$ inch, and that greater precision can only be attained by closer centring.

We may now apply the criterion of negligibility of § 20 to ascertain first for what length of traverse line, L, a sighting-and-reading error of $\pm$ 12 seconds ($5{\cdot}8 \times 10^{-5}$ radians) may be neglected when $\gamma = 0{\cdot}024$ foot and $T = 180°$. The criterion gives—

$$v \leqq \frac{1}{3}\left\{\frac{4r}{\pi L}\right\},$$

* The maximum displacement is equivalent to an average eccentricity (r) of $\frac{2}{3}$ 0·024 foot—*i.e.*, 0·016 foot, or $\frac{3}{16}$ inch.

or L $\leqq$ 114 feet as the result required. A further justification, or, properly, an excellent excuse, may here be deduced for taking v as constant for all values of L ; or, in other words, for drawing D (Fig. 16) as a straight line ; for though v, strictly speaking, must increase to some extent as L becomes shorter than 100 feet, it cannot do so at the same rate that the curve shows the centring error to increase. Even if v were to become 60 seconds for traverse lines 24 feet long, or five times the amount selected in constructing the curves, it would, nevertheless, be almost negligible under the above criterion when $\gamma = 0{\cdot}024$ foot and T = 180°. It would, therefore, be of no immediate consequence to determine —were it possible—the rate of increase of v as the lines become very short, since the increase can have very little, if any, sensible effect.

The same criterion also allows of a test being applied on the legitimacy of the assumption, already made, to the effect that triangulation lines are usually of such a length that a small centring displacement has no tangible influence. An average sighting-and-reading error of $\pm$ 12 seconds will be reduced to $\pm \frac{12}{\sqrt{3}}$, or $\pm$ 7 seconds, by three reiterations. Consider the case of two sides of a triangle of equal length, L, embracing an angle of about 60°. If $\gamma = 0{\cdot}024$ foot, L will be such that the average angular centring error is equal to or less than one-third of 7 seconds (expressed in radians) when L $\geqq$ about 900 feet. As the sides of the main triangles in a minor triangulation scheme generally exceed 300 yards in length, and, moreover, since it is usual—though evidently not strictly necessary—to centre with greater care in triangulation than in traversing, we may safely conclude that centring errors in triangulation are negligible in their effects when compared with those we have termed sighting-and-reading errors.

Therefore, we are justified in assuming, as we shall do when discussing the influence of errors in triangulation, that the average error in each of the angles of a triangle is of the same magnitude, being, in point of fact, equal to v, the sighting-and-reading error.

33. **Transference of Bearing from Reference Lines.**—It is good practice in minor triangulation or surface traversing to make use of one or more reference lines, half a mile or more in length, as the means of orienting the survey. A side of a triangulation triangle may often be used as reference for a traverse, but when that is not available, or in the case of the reference of a triangulation system itself, the true bearing of the line chosen has to be found directly by the aid of observations on the sun or a star; then, by turning off the angle between the reference and the first line of the survey, the bearing of the latter line is obtained straightway. In some cases a long reference is used as false meridian, and no attempt is made to ascertain its bearing: the lines of the survey are in that case referred to the direction of the reference as arbitrary north.

If L be the length of a line of the survey, which has one end in common with one end of the reference, we may inquire what error is likely to result in transferring the bearing from the reference to this survey line.

The reference will in general be many times longer than L; therefore, its reciprocal will be negligible as compared with $1 \div L$; hence (63) reduces, in this particular case, to—

$$\textit{Average angular centring error} = \pm \frac{4\gamma}{3\pi L}.$$

Combining this with v, the sighting-and-reading error, we get—

$$\left.\begin{array}{l}\textit{Average angular error in transferring} \\ \textit{bearing from a reference to a survey} \\ \textit{line having one end in common}\end{array}\right\} = \pm \sqrt{\left\{v^2 + \frac{16\gamma^2}{9\pi^2 L^2}\right\}}. \quad (69)$$

In triangulations, as has already been proved, the centring error (namely, the second term in (69)) is usually negligible. Eq. (69) is made use of in Problem IV., Chapter V. It shows that the error is independent of the angle between the reference and the survey line.

34. **Errors of Bearing in Simple Compass Traverses.**—When some form of compass instrument, such as the miner's dial or the prismatic compass, is used as the means of measuring bearings, and the needle is read to obtain them, the magnetic bearing of any line is ascertained independently of that of any other. A further simplification results from the absence of centring error, since, in this simplest of all traversing methods, the instrument is set, not at every station, but at alternate stations. It therefore follows that if $\pm u$ be the average error made in obtaining the bearing of a line, the bearing of all lines of the traverse can be considered as being affected alike by this average deviation u. One exception to this statement, however, needs to be noted, and is, that in a traverse in which some of the lines are exceptionally short, the average error in bearing of the very short lines is likely to be greater than that of ordinary sights.

35. **Errors of Bearing in Theodolite Traverses.**—In ordinary theodolite traverses, where the instrument is set at every station, and in which *angles* are measured, the precision attained in the bearing of any line is dependent on that of the preceding line ; hence, on the average, the error in the bearing of the nth line is greater than that of any of the lines behind it, being, indeed, compounded of the errors in all the preceding lines.

Consider the case of a theodolite traverse of n lines, of which the first has a known bearing, β_1.

In practice β_1 is determined in a variety of ways ; it may be a true bearing or a magnetic, or an arbitrary one. Only when the first line of the traverse is used as "false meridian" will the initial bearing—in this case zero—be affected by no error; in all other cases it will be influenced by an error of greater or lesser magnitude. Although sometimes difficult to assess, this initial error is always easy to apply, since it will swing the survey, as a whole, about the first point as pivot, either to one side or to the other. Hence, if we term x_1 the average amount by which the end-point of the traverse is swung by the initial error in β_1, and x_2 the average error in position of the same point due to imperfections in the survey itself, then, no matter what may be the relative clinures of these vectors, their sum, R, can be obtained by relation (30), thus—

$$R = \pm \sqrt{x_1^2 + x_2^2}. \qquad . \qquad . \qquad (70)$$

At present we are concerned with the second of these components; its magnitude is independent of that of the first; hence in determining x_2 we may assume x_1 as non-existent. In other words, we may proceed with the investigation on the assumption that β_1 is without error, remembering that (70) permits of the initial error in bearing being taken into account after x_2 has been evaluated.

Let L_1, L_2, . . . L_n represent the lengths of the lines of the traverse under consideration; T_1, T_2 . . . T_{n-1}, the traverse angles; t_1, t_2 . . . t_{n-1}, the average errors by which those angles are respectively affected; also let β_1, β_2 . . . β_n be the bearings of the lines, and β'_1, β'_2 . . . β'_n the average errors in those bearings, β'_1 being taken as zero.

Then the bearing β_m, of any line, is given by—

$$\beta_m = \beta_1 + T_1 + T_2 \ . \ . \ . + T_{m-1} - 180^\circ(m-1),$$

$$\text{and} \quad \beta'_m = \pm\sqrt{\{t_1^2 + t_2^2 \ . \ . \ . + t_{m-1}^2\}}. \qquad . \qquad . \qquad . \qquad (71)$$

From (68) we have that the average error in any angle—

$$t_m = \pm\sqrt{\left\{v^2 + \frac{16\,\gamma^2}{9\,\pi^2}\left(\frac{1}{L_m^2} + \frac{1}{L_{m+1}^2} - \frac{2\cos T_m}{L_m L_{m+1}}\right)\right\}}. \quad (72)$$

Combining expressions (71) and (72)—

$$\beta'_m = \pm\sqrt{\left\{(m-1)\,v^2 + \frac{16\,\gamma^2}{9\,\pi^2}\left(\frac{1}{L_1^2} + \frac{2}{L_2^2} + \frac{2}{L_3^2} \cdots + \frac{2}{L_{m-1}^2} + \frac{1}{L_m^2}\right) - \frac{32\,\gamma^2}{9\,\pi}\left(\frac{\cos T_1}{L_1 L_2} + \frac{\cos T_2}{L_2 L_3} \cdots + \frac{\cos T_{m-1}}{L_{m-1} L_m}\right)\right\}} \quad . \quad . \quad (73)$$

This may also be written—

$$\beta'_m = \pm\sqrt{\left\{\beta'^2_{m-1} + v^2 + \frac{16\,\gamma^2}{9\,\pi^2}\left(\frac{1}{L_{m-1}^2} + \frac{1}{L_m^2} - \frac{2\cos T_{m-1}}{L_{m-1} L_m}\right)\right\}} \quad (74)$$

The latter form is of more service in studying in detail the propagation of angular error in an actual traverse, since the average error of the bearings are then worked out in succession, commencing with β'_2.

By means of equations (73) or (74) the average error in summation of the angles of a closed traverse, or polygon, can be determined.

36. **Average Total Error at the End of the *n*th Line of a Simple Compass Traverse.**—Consider the case of a traverse in which bearings are taken by a compass and lengths by an ordinary chain. Let $\pm u$ be the average error in each bearing, and K_1 the value K of eq. (59) assumes for a chain. Let L_m be any line of the traverse, and l_m the average error in its length; then, from (59) we have—

$$l_m = \pm K_1 \sqrt{L_m}. \quad . \quad . \quad . \quad (75)$$

Owing to the error in bearing, the end of any line L_m will be displaced by the average amount $\pm L_m u$. Therefore the displacement of the point will be due to the resultant of the average vector errors l_m and $L_m u$—that is to say, it will be $\pm\sqrt{K_1^2 L_m + L_m^2 u^2}$, by eq. (30).

Now, this average displacement will be passed on to the next line of the traverse, and, indeed, to all subsequent lines; hence the end-point of the traverse will be affected by an average error compounded of all such errors as $\sqrt{K_1^2 L_m + L_m u^2}$. Thus, by (30)—

Average total error at the end of the compass traverse

$$= \pm \sqrt{\{K_1^2(L_1 + L_2 \ldots + L_n) + u^2(L_1^2 + L_2^2 \ldots + L_n^2)\}}. \quad (76)$$

If the total length of the traverse—namely, $(L_1 + L_2 \ldots + L_n)$, is constant, a choice can sometimes be made between performing the work by a few long lines or by a greater number of shorter ones. Now, the magnitude of $(L_1^2 + L_2^2 \ldots + L_n^2)$ diminishes as the number of lines increases; therefore, in simple compass traversing, numerous short lines are preferable to few long ones when the average total error is the question of chief importance. This fact is pretty well known, but the graphical proof sometimes attempted, though plausible at first sight, is unsound and incapable of withstanding careful scrutiny.

37. **Average Total Error at the End of the nth Line of a Theodolite Traverse.**—By means of relation (74) the average errors in the bearings of the traverse lines are successively ascertainable: similarly, the average errors affecting the lengths may be determined from (59), writing K_2 in place of K, where K_2 is the value the coefficient assumes for a steel tape. Using the same notation as before, we have—

Average total error at the end of the theodolite traverse

$$= \pm \sqrt{\{K_2^2(L_1 + L_2 \ldots + L_n) + (L_2^2\beta_2'^2 + L_3^2\beta_3'^2 \ldots + L_n^2\beta_n'^2)\}}. \quad (77)$$

The total error may be analysed in the following way:—

The co-ordinates of the end point of the traverse, with reference to the first point as origin, are—

$$\textit{Latitude} = L_1 \cos\beta_1 + L_2 \cos\beta_2 \ldots + L_n \cos\beta_n;$$

and $\textit{Departure} = L_1 \sin\beta_1 + L_2 \sin\beta_2 \ldots + L_n \sin\beta_n$.

Therefore, by a direct application of eq. (13) we obtain—

Average error in the latitude of the end point

$$= \pm \sqrt{\{K_2^2(L_1 \cos^2 \beta_1 + L_2 \cos^2 \beta_2 \ldots + L_n \cos^2 \beta_n) + (L_2^2\beta_2'^2 \sin^2 \beta_2 + L_3^2\beta_3'^2 \sin^2 \beta_3 \ldots + L_n^2\beta_n'^2 \sin^2 \beta_n)\}} \quad (78)$$

Average error in the departure of the end point

$$= \pm \sqrt{\{K_2^2(L_1 \sin^2 \beta_1 + L_2 \sin^2 \beta_2 \ldots + L_n \sin^2 \beta_n) + (L_2^2\beta_2'^2 \cos^2 \beta_2 \ldots + L_n^2\beta_n'^2 \cos^2 \beta_n)\}} \quad (79)$$

It is seen that (77) may be obtained by compounding (78) and (79).

Of these last three equations (77) is the simplest and most useful; it allows of a valuation being made of the final error which might reasonably be expected in any theodolite traverse, whether "closed" or "open." Relations (78) and (79) are of value when attention is directed to the error in some particular direction, for the co-ordinate axes can purposely be arranged so that the direction in question is either north-and-south, or east-and-west, with reference to them.

It follows from (77), since $\beta'_2, \beta'_3 \ldots \beta'_n$ all attain their maximum values when $T_1, T_2, T_3 \ldots T_n$ are all 180°, that, other things being equal, a *straight traverse* will be affected with a greater average total error than any other traverse with the same number and lengths of lines.

38. **The Relative Accuracy of Compass and Theodolite Traversing.**—Consider a straight traverse of n equal lines, running between two fixed points F feet apart. In these circumstances $n\,L = F$ (L being the length of any of the lines), and $T_1 = T_2 \ldots = T_{n-1} = 180°$. Therefore, (76) reduces to—

Average total error in a simple compass traverse

$$= \pm \sqrt{\{K_1^2 F + u^2 L F\}} \quad (80)$$

In like manner (77) reduces to—

Average total error in a theodolite traverse

$$= \pm \sqrt{\left\{ K_2^2 F + \frac{F(F - L)}{2} \left(v^2 + \frac{64\gamma^2}{9\pi^2 L^2} \right) \right\}}. \quad (81)$$

Now, while (80) diminishes as L diminishes, (81) increases as L diminishes; thus, keeping F constant, it follows that there must be some value of L for which the average total error of the theodolite traverse equals the average total error of the compass traverse, and this condition will be secured when—

$$K_1^2 F + u^2 LF = K_2^2 F + \frac{F(F - L)}{2} \left(v^2 + \frac{64\gamma^2}{9\pi^2 L^2} \right).$$

In these circumstances L will be short, and unless the fixed points between which the traverses run are very close together, it is sufficiently exact to assume (F − L) as equal to F; when the above simplifies to the following cubic equation :—

$$L^3 + L^2 \left\{ \frac{2(K_1^2 - K_2^2) - F v^2}{2 u^2} \right\} - \frac{32\gamma^2}{9\pi^2 u^2} = 0. \quad (82)$$

To illustrate the value of this result, let us make a comparison between the miner's dial (used as a simple compass instrument) and a 5-inch theodolite, our experience with the instruments having led us to conclude that K_1 (for the 100-foot chain) = 0·0096, K_2 (for the 100-foot steel band) = 0·0063, $v = \pm 12$ seconds, and $u = \pm \frac{1}{8}°$. Instead of taking $\gamma = 0·024$ foot, as has been done in former examples, we may profit by the results obtained in an earlier part of the chapter, and now that we are dealing with exceptionally short lines, reduce the maximum centring displacement to 0·016 foot, or $\frac{3}{16}$ inch. Substituting these values in (82), and taking F, the total length of the traverse as 1,000 feet, L is found to be approximately 23 feet. In other words, under the stated conditions, if the lines of a straight traverse, all assumed equal, were more than 23 feet long, the theodolite would be expected to give more accurate results than the miner's dial; yet if they were shorter than this figure the compass instrument would have the advantage, providing local magnetic attraction were absent. It also follows from (82) that this limiting length of line increases as F increases—that is to say, the *relative* accuracy of the two modes of traversing depends on the total length of the survey, being more in favour of the compass method with long traverses than with short ones.

Before leaving this section of the subject the author desires to make it quite clear that such a result as that just obtained (viz., L = 23 feet) can only be a very rough one even under the stated conditions, and must by no means be taken as being generally correct. In this instance it is the method illustrated which possesses value, rather than the result derived. There are so many factors to be taken into account that it is not possible to discuss the general case completely. In the above example we have, if anything, favoured the theodolite in assuming β'_1, the error in bearing of the first line of the traverse, as being zero. Were a value to be taken for β'_1, eq. (70) would need to be applied to include its effect, F β'_1 being written instead of x_1, and the "total error" from (81) instead of x_2. Now, it is evidently quite impossible to give any specific value to β'_1 such as would serve, even approximately, as a general average; the magnitude of β'_1 depends on things at present outside our knowledge—namely, on what occurred prior to the commencement of the traverses under discussion. Questions such as the following immediately arise in reference to this initial error: Does the theodolite traverse commence from a triangulation station and become oriented by means of a preliminary sight over a triangulation line? If so, may it be assumed that the triangulation was conducted with such a degree of accuracy that the error in the bearing of the reference may be assumed negligible under the criterion of § 20 ?—in which case β_1 would equal

$$\pm \sqrt{\left\{ v^2 + \frac{16\,\gamma^2}{9\,\pi^2\,L_1{}^2} \right\}},$$

from eq. (69). Or, was the traverse oriented by taking the magnetic bearing of the first line, using, say, the trough compass ?—for in that case the effect of this initial error β_1 may be so large as to exceed considerably that accruing within the traverse.

These, among other possible cases, will serve to show the difficulty of dealing with the problem in a general way; yet, in making a comparison of the compass and theodolite under any actually existing set of conditions, all questions such as the above are quickly answered.

When the comparison is made between a theodolite traverse referred to the true meridian and a compass traverse referred to the magnetic meridian, another factor of the first importance enters—namely, the accuracy by which the magnetic meridian has been determined. In the case where the compass can be sighted over a line—such as a triangulation line—whose true bearing has been carefully ascertained, the declination may be taken as being determined with an average error of $\pm u$; the average total error in the compass traverse must then be adjusted by means of (70), x_1 being F u, and x_2 the value given by (80).

In short, while it is a matter of the greatest difficulty to make any serviceable *general* comparison between the accuracy of compass and theodolite traversing, it is not at all so difficult to make a comparison to suit any set of conditions which may arise in practice.

CHAPTER V.

THE APPLICATION OF THE METHODS OF DETERMINING AVERAGE ERROR TO CERTAIN PROBLEMS IN TRAVERSING.

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THE APPLICATION OF THE METHODS OF DETERMINING AVERAGE ERROR TO CERTAIN PROBLEMS IN TRAVERSING.

39. The results obtained in the last chapter are here made use of in dealing with certain problems in practical surveying.

PROBLEM I.—*The following is an actual closed traverse conducted by means of a 5-inch theodolite and a 100-foot chain. Experience with the instruments had shown that the average sighting-and-reading error* (v) *could be taken as* $\pm 12''$ *for the instrumental method used, and the coefficient.* K_1, *for the chain as* 0·0096. *Taking the maximum permissible centring displacement* (γ) *as* 0·024 *foot, it is desired to find the average error of closure, so as to be able to compare the actual error of closure with it.*

Fig. 17 shows the traverse, plotted to the scale of 400 feet to the inch.

Lines.	Angles.	Bearings.	Lengths.	Reduced Latitude. (Calculated.)	Reduced Departure. (Calculated.)
A B,		N.	546·3 ft.	+ 546·30	0
	179° 47′ 10″				
B C,		N. 0° 12′ 50″ W.	680·8 „	+1127·10	− 2·54
	99° 09′ 30″				
C D,		N. 81° 03′ 20″ W	1120·3 „	+1401·28	−1109·22
	96° 01′ 40″				
D E,		S. 14° 58′ 20″ W.	1388·8 „	+ 59·60	−1468·02
	56° 15′ 25″				
E F,		N. 71° 13′ 45″ E.	588·1 „	+ 248·84	− 911·19
	214° 01′ 20″				
F A,		S. 74° 44′ 55″ E.	944·1 „	+ 0·51	− 0·33
	74° 44′ 50″				

Actual error in angles of polygon = 5″.
Actual error of closure = $\sqrt{(0{\cdot}51)^2 + (0{\cdot}33)^2} = 0{\cdot}61$ ft.

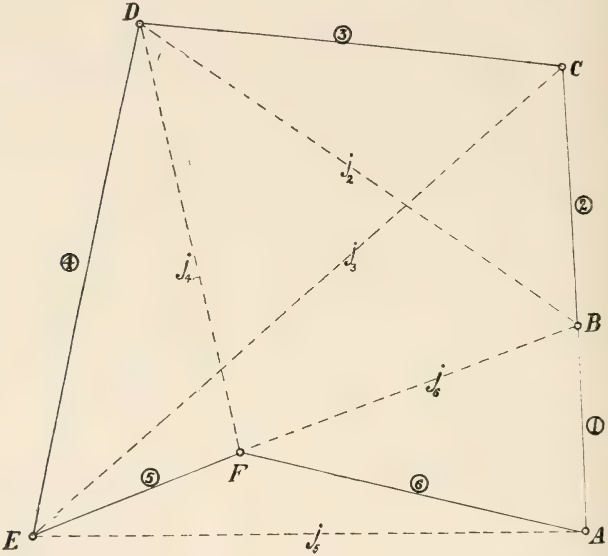

Fig. 17.

Method of Solution.—The first step is to determine the average angular error in each traverse angle due to centring displacement. Leaving the application of transectors to a later problem, we shall here directly apply eq. (63). For the present case this may be written—

Square of average angular centring error

$$= \left(\frac{4 \times 0{\cdot}024}{3 \times 3{\cdot}1416}\right)^2 \left\{\frac{1}{L_1^2} + \frac{1}{L_2^2} - \frac{2 \cos T}{L_1 L_2}\right\}$$

$$= 10^{-5} \times 9{\cdot}89 \left\{\frac{1}{L_1^2} + \frac{1}{L_2^2} - \frac{2 \cos T}{L_1 L_2}\right\}.$$

With regard to the first traverse angle, namely, that at B, T in that case is seen to be so nearly equal to 180° that cos T may be taken as − 1 ; hence we have—

$$\left.\begin{matrix}\textit{Square of centring error} \\ \textit{affecting B}\end{matrix}\right\} = 10^{-5} \times 9{\cdot}89\left\{\frac{1}{550} + \frac{1}{1120}\right\}^2.$$

Taking the reciprocals from Table I. of the Appendix, this last expression equals—

$$B_1^2 = 10^{-5} \times 9{\cdot}89(10^{-3} \times 1{\cdot}82 + 10^{-3} \times 0{\cdot}91)^2 = 10^{-9} \times 1{\cdot}077.$$

Similarly,

Square of centring error affecting C

$$= 10^{-5} \times 9{\cdot}89\left\{\frac{1}{1120^2} + \frac{1}{1390^2} + \frac{2 \cos 81}{1120 \times 1390}\right\},$$

or $C_1^2 = 10^{-10} \times 3{\cdot}33$.

In like manner—

$$D_1^2 = 10^{-10} \times 1{\cdot}43,$$
$$E_1^2 = 10^{-10} \times 2{\cdot}00,$$
$$F_1^2 = 10^{-10} \times 6{\cdot}92,$$
$$A_1^2 = 10^{-10} \times 3{\cdot}39.$$

The second step is to ascertain the average error in the traverse angles. This is done by applying equation (68).

$v = \pm 12$ secs. $= \pm 10^{-5} \times 5{\cdot}819$ radians (from Table II.)

$$v^2 = 10^{-9} \times 3{\cdot}386 \text{ (Table II.).}$$

Terming the average errors in the respective traverse angles t_B, t_C, etc., we have—

$$t_B^2 = v^2 + B_1^2 = 10^{-9} \times 3{\cdot}39 + 10^{-9} \times 1{\cdot}08 = 10^{-9} \times 4{\cdot}47$$
$$t_C^2 = v^2 + C_1^2 = 10^{-9} \times 3{\cdot}39 + 10^{-9} \times 0{\cdot}33 = 10^{-9} \times 3{\cdot}72$$
$$t_D^2 = v^2 + D_1^2 = 10^{-9} \times 3{\cdot}39 + 10^{-9} \times 0{\cdot}14 = 10^{-9} \times 3{\cdot}53$$
$$t_E^2 = v^2 + E_1^2 = 10^{-9}(3{\cdot}39 + 0{\cdot}20) = 10^{-9} \times 3{\cdot}59$$
$$t_F^2 = v^2 + F_1^2 = 10^{-9}(3{\cdot}39 + 0{\cdot}69) = 10^{-9} \times 4{\cdot}08$$
$$t_A^2 = v^2 + A_1^2 = 10^{-9}(3{\cdot}39 + 0{\cdot}34) = 10^{-9} \times 3{\cdot}73.$$

We may now apply (74) to find the average errors in the bearings of the lines. As line A B is taken as false meridian, $\beta'_1 = 0$. The error in the bearing of B C is termed β'_2; that in C D, β'_3, and so on :—

$$\beta'^2_2 = 10^{-9} \times 4{\cdot}47$$
$$\beta'^2_3 = 10^{-9}(\ 4{\cdot}47 + 3{\cdot}72) = 10^{-9} \times \ \ 8{\cdot}19$$
$$\beta'^2_4 = 10^{-9}(\ 8{\cdot}19 + 3{\cdot}53) = 10^{-9} \times 11{\cdot}72$$
$$\beta'^2_5 = 10^{-9}(11{\cdot}72 + 3{\cdot}59) = 10^{-9} \times 15{\cdot}31$$
$$\beta'^2_6 = 10^{-9}(15{\cdot}31 + 4{\cdot}08) = 10^{-9} \times 19{\cdot}39$$
$$\beta'^2_7 = 10^{-9}(19{\cdot}39 + 3{\cdot}73) = 10^{-9} \times 23{\cdot}12$$

By β'_7 is meant the error in bearing of line A B, as ascertained by working round the polygon and making use of all the traverse angles ; in other words, it is the average summation error of the angles. From above, $\beta_7' = 10^{-4}\sqrt{2{\cdot}31}$ radians, or—

Average total error in angles of the polygon = 31 secs.

The last step consists of the application of (77), thus :—

$$K_1 = 0{\cdot}0096,\ \Sigma(L) = 5268 \text{ feet.}$$
$$\therefore \qquad K_1^2\,\Sigma(L) = 0{\cdot}486 \text{ foot.}$$
$$L_2^2\,\beta'^2_2 = \ \ 680^2 \times 10^{-9} \times \ \ 4{\cdot}47 = 0{\cdot}0021$$
$$L_3^2\,\beta'^2_3 = 1120^2 \times 10^{-9} \times \ \ 8{\cdot}19 = 0{\cdot}0103$$
$$L_4^2\,\beta'^2_4 = 1390^2 \times 10^{-9} \times 11{\cdot}72 = 0{\cdot}0226$$
$$L_5^2\,\beta'^2_5 = \ \ 590^2 \times 10^{-9} \times 15{\cdot}31 = 0{\cdot}0053$$
$$L_6^2\,\beta'^2_6 = \ \ 940^2 \times 10^{-9} \times 19{\cdot}39 = 0{\cdot}0171$$

Total, 0·0574

Average error of closure = $\sqrt{\{0{\cdot}486 + 0{\cdot}057\}} = 0{\cdot}74$ ft.

Of the two quantities under the last root sign, the first is due to errors in the linear, and the second to errors in the angular measurements. Expressed as a fraction of the total length of the traverse, the average error of closure is about 1 in 7,000. The traverse is a very favourable one for a high degree of accuracy, the lines being exceptionally long. The actual closing error, being 0·61 foot, is 0·82 of the average.

A glance through the above calculation will show—

(*a*) That a considerably greater centring displacement could have been allowed without materially affecting the resulting accuracy;

(*b*) That a less precise mode of measuring the angles—say, reading one vernier instead of both for every sight—could have been used without noticeable effect;

(*c*) If greater precision had been required, it could best have been obtained by improving the linear rather than the angular measurements—*i.e.*, by substituting the steel tape for the chain.

PROBLEM II.—*For the purpose of illustrating the effect of using shorter lines, it is now required to determine the average closing error in a traverse having the same angles, but with each line one-tenth of the length of the corresponding line of Problem I.*

Fig. 17 will again represent the traverse, but the scale will now be 40 feet to the inch. The lengths of the lines will be as follows :—

A B, . . .	55 feet.	
B C, ...	68 ,,	
C D, ...	112 ,,	
D E, . . .	139 ,,	Σ (L) = 527 feet.
E F, ...	59 ,,	
F A, ...	94 ,,	

By making the scale drawing (Fig. 17) by the aid of a protractor, the transectors can be ascertained. The transector crossing the angle B is termed j_1; that crossing angle C, j_2, and so on. The lengths of the transectors, as taken from the plan, are as follows :—

j_1, ...	123 feet.
j_2, ...	141 ,,
j_3, ...	187 ,,
j_4, ...	116 ,,
j_5, ...	146 ,,

Taking $\gamma = 0{\cdot}024$ foot, as before, we have, from (66)—

Square of angular error due to centring, angle B $= 10^{-5} \times 9{\cdot}89\left(\frac{123}{55 \times 68}\right)^2 = 10^{-7} \times 1{\cdot}069.$

Ditto, angle C $= 10^{-5} \times 9{\cdot}89\left(\frac{141}{68 \times 112}\right)^2 = 10^{-8} \times 3{\cdot}39.$

The others are obtained similarly, and are stated in the columns below. As the use of the transectors is the only new point here illustrated, the various steps of the rest of the calculation may be more compactly set in tabular form. The numbers in brackets refer to the equations from which the columns are computed :—

Angles.	Square of Centring Error.	t^2 (68).	β'^2 (74).	$L^2 \beta'^2$
A B C	$10^{-7} \times 1{\cdot}07$	$10^{-8} \times 11{\cdot}03$	$10^{-8} \times 11{\cdot}03$	0·0005
B C D	$10^{-8} \times 3{\cdot}39$	$10^{-8} \times 3{\cdot}73$	$10^{-8} \times 14{\cdot}76$	0·0018
C D E	$10^{-8} \times 1{\cdot}43$	$10^{-8} \times 1{\cdot}77$	$10^{-8} \times 16{\cdot}53$	0·0032
D E F	$10^{-8} \times 1{\cdot}98$	$10^{-8} \times 2{\cdot}32$	$10^{-8} \times 18{\cdot}85$	0·0007
E F A	$10^{-8} \times 6{\cdot}85$	$10^{-8} \times 7{\cdot}19$	$10^{-8} \times 26{\cdot}04$	0·0023
			Total,	0·0085

$$K_1^2 \, \Sigma \, (L) = 527 \, (0{\cdot}0096)^2 = 0{\cdot}0486.$$

Applying (77), we obtain—

$$\textit{Average error of closure} = \sqrt{(0{\cdot}0486 + 0{\cdot}0085)}$$
$$= 0{\cdot}24 \text{ foot.}$$

That is, about 1 in 2,200.

A considerably less precision would, therefore, be expected with these shorter lines than with those whose lengths are given in Problem I.

Problem III.—*When a straight road has to be driven between two points underground, an accurate survey becomes necessary to determine the bearing of the road. It is usual to drive such a road from both ends simultaneously, and, as the drifts must hole as exactly as possible, a survey of this kind forms a trying problem to the mine surveyor.*

The following example is inserted to show the practical help afforded by our processes of analysis in such an instance. It is also intended to illustrate .—

(*a*) The use of reciprocal transectors, and

(*b*) The mode of assessing the average error arising *in a given direction.*

In the case shown in Fig. 18 (scale 200 feet to the inch), a straight road is to be driven to connect A and B. A preliminary underground inspection serves to locate the positions of the theodolite stations, and by taking a chain underground also, measurements can be made so as to allow of these points being indicated on the mine plan with a precision sufficient for the purposes of analysis. We assume that the small circles indicate the stations on the mine plan, located in this manner.

We desire to find the accuracy by which the angles of the traverse must be measured in order to be reasonably sure *that the error in the bearing of* A B, *when it comes to be computed from the results of the theodolite traverse, will not exceed two minutes of arc.*

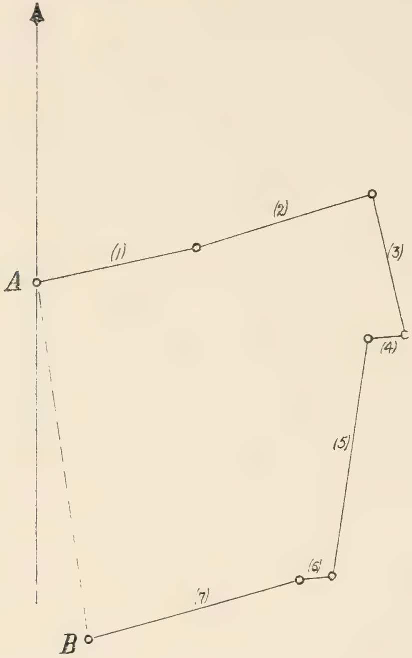

Fig. 18.

All points are marked in the roof of the mine ; therefore the maximum centring displacement can be taken as $\frac{1}{8}$ inch. The steel tape will be used, and K_2 can be taken as 0·0063.

Method of Calculation.—The steps of the calculation may now be stated :—

(1) The lengths of the lines are first scaled off from the mine plan (column 2, below).

(2) The angles of the closed figure are obtained by means of a protractor (column 4, below)

(3) Since Line 1 is to be used as a fixed reference-direction, it has no error of bearing. We desire to ascertain by how much the average displacement of B will swing the bearing of A B. Only the component of this

Fig. 19.

displacement at right angles to A B will affect the bearing of that line. Hence, if we take A B as false meridian for the purposes of the calculation, we shall need to find the average total error in *departure* of B. The third step is, therefore, to express the bearings of the lines with regard to A B as arbitrary meridian (column 5, below). [Properly, we ought to say that a direction making a fixed angle (in this case, 94°) with the fixed direction of Line I is taken as false meridian, so that this meridian shall coincide as nearly as possible with A B].

(4) Having obtained the reciprocals of the lengths of the traverse lines from Table I. of the Appendix, these are now plotted along with the traverse angles to form the "reciprocal traverse" (Fig. 19)—*Scale*, 1 inch equals 0·012).

(5) The reciprocal transectors are measured from the reciprocal traverse (column 6).

(6) The average angular centring errors are calculated by the aid of (67), and compounded with v to obtain the average errors in the traverse angles. The squares of these values are stated in column 7. The sighting-and-reading error, v, is the unknown quantity in the calculation.

(7) The squares of the average errors in the bearings of the lines are found by means of (74); these are stated in column 8.

(8) Equation (79) is now made use of to evaluate the average error in the departure of B. The remainder of the columns below are utilised in deriving this final result.

But $862\ K_2^2 = 0{\cdot}0342$; therefore, by eq. (79)—

Average error in departure of B

$$= \pm \sqrt{\{0{\cdot}0342 + 0{\cdot}0019 + 448057\, v^2\}}, \quad . \quad (83)$$

Now, 2′ of arc would be equivalent to a swing of 0·279 foot in the length of A B.

By equating these last results, thus :—

$$\sqrt{(0{\cdot}0361 + 448057\, v^2)} = 0{\cdot}279,$$

we find, after converting radians into seconds, that $v = \pm 64$ seconds.

That is to say, if the average sighting-and-reading error during the theodolite traverse were about 1 minute,

1	2	3	4	5	6	7	8	9	10	11	12
Lines.	Lengths Feet, L.	Reciprocals, $\frac{1}{L}$.	Angles.	Bearings, β.	Reciprocal Transsectors, ω.	v^2.	β'^2.	Log $\cos^2\beta$	Log $\sin^2\beta$	L $\sin^2\beta$.	$L^2\cos^2\beta\cdot\beta'^2$.
1	216	0·00465		266°			0	$\bar{3}$·6872	1·9979	215·00	...
			175°		0·0083	$10^{-9}\times 1·52+v^2$					
2	245	0·00408		261°			$10^{-9}\times 1·52+v^2$	$\bar{2}$·3887	1·9892	240·00	$10^{-6}\times 2·24$ + 1,469 v^2
			273°		0·0065	$10^{-10}\times 8·26+v^2$					
3	190	0·00526		354°			$10^{-9}\times 2·35+2v^2$	1·9952	2·0385	2·03	$10^{-5}\times 8·39$ + 71,440 v^2
			278°		0·0201	$10^{-9}\times 7·90+v^2$					
4	50	0·0200		92°			$10^{-9}\times 10·25+3v^2$	$\bar{3}$·0856	$\bar{1}$·9995	49·95	$10^{-8}\times 3·12$ + 9 v^2
			105°		0·0211	$10^{-9}\times 8·71+v^2$					
5	317	0·00318		17°			$10^{-9}\times 18·96+4v^2$	1·9612	2·9319	27·10	$10^{-3}\times 1·743$ + 367,640 v^2
			255°		0·0248	$10^{-8}\times 1·20+v^2$					
6	42	0·0238		92°			$10^{-9}\times 30·96+5v^2$	$\bar{3}$·0856	$\bar{1}$·9995	42·00	$10^{-8}\times 7·05$ + 11 v^2
			171°		0·0272	$10^{-8}\times 1·45+v^2$					
7	290	0·00345		83°			$10^{-9}\times 45·5+6v^2$	2·1718	1·9935	285·70	$10^{-5}\times 5·683$ + 7,488 v^2
			83°								
B A	480	...		180°							
									Totals,	861·78	$10^{-3}\times 1·89$ + 448,057 v^2

the average error in bearing of A B, as ascertained from the results of the traverse, would be 2 minutes.

Reference to Table III. of the Appendix shows that if an instrumental method were adopted which gave a value of v about $\pm$ 1 minute, the odds would be roughly 4 to 3 in favour of the *actual* error in bearing of A B being less than 2 minutes.

On the other hand, if we were to use a method in which v is brought down to $\pm$ 15 seconds, we find from (83) that the average error in departure of B would equal $\pm$ 0·196 foot ; in which case a swing of 0·279 foot would be 1·42 times the average error, and thus we should secure a 3 to 1 chance of the actual error in the bearing of A B being less than 2 minutes (*see* Table III.).

PROBLEM IV.—*This problem is on the well-known operation of transferring bearing underground by means of two vertical shafts. It is designed to show how the methods of the last chapter can be used to ascertain the accuracy of this mode of shaft-connection, and also to illustrate the manner of finding the average error in taking bearing from a long reference line.*

Fig. 20 shows the surface and underground traverses (*scale*, 120 feet to the inch), and the data are as follows :—

Five sun observations were taken to determine the bearing of the reference, A R, and the results were :— 40° 30′ 45″, 40° 31′ 15″, 40° 31′ 30″, 40° 30′ 10″, 40° 30′ 00″. The mean of these values is 40° 30′ 44″, and the residual errors for the five observations are respectively 1″, 31″, 46″, 34″, and 44″. These give the apparent average error of a single observation as $\pm$ 31″. The real average error of the mean is, therefore, $31 \div \sqrt{5-1} = \pm 15''$ (*see* eq. (10)).

The surface and underground traverses were conducted by the same instrument, and the average sighting-and-reading error, v, may be taken as $\pm$ 15″. The steel tape was used in both surveys, and the coefficient, K_2, may be taken as 0·0063. The maximum displacement in centre permitted at the surface was ¼ inch, and underground ⅛ inch.

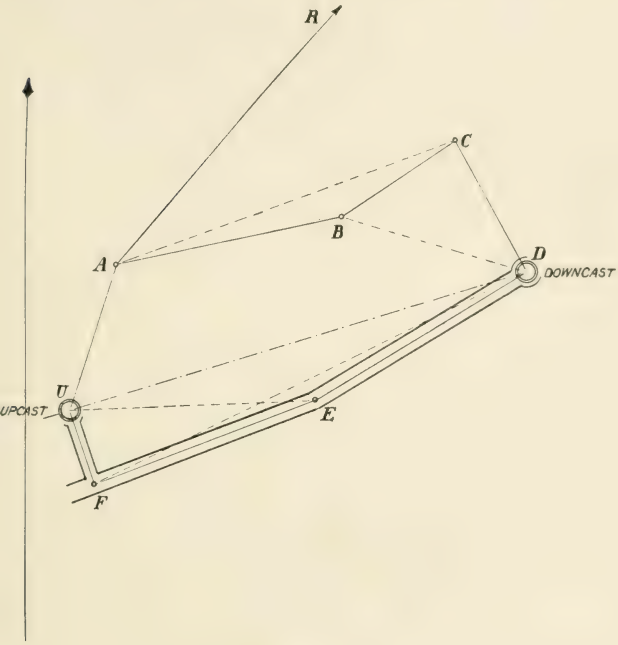

Fig. 20.

The following were the lengths of the lines and the values of the traverse angles, stated, the former to single feet, and the latter to half degrees, this being close enough for our present purpose :—

Surface Traverse—

U A,	. 118 feet.	Angle	R A U,	157° 30′
A B,	. 179 ,,	,,	R A B,	37° 30′
B C,	. 106 ,,	,,	A B C,	158° 00′
C D,	. 116 ,,	,,	B C D,	275° 30′

Underground Traverse—

U F,	. 60 feet.	Angle	U F E,	87° 00′
F E,	. 183 ,,	,,	F E D,	170° 00′
E D,	. 192 ,,			

These data allow protractor-drafts of the surveys to be prepared, and fitted together as in Fig. 20. From the diagram the following additional information is obtained :

Approximate bearing of U D, . . . 73° 30′
Approximate length, U D, . . . 369 feet.

Angle D U F, 89° ; angle E D U, 14° ; angle A U D, 55° ; angle C D U, 78° 30′.

The transectors having been drawn, their lengths are found to be :—

A C, 280 ft. ; B D, 149 ft. ; D F, 373 ft. ; U E, 190 ft.

Method of Calculation.—The reference line being very long as compared with U A or A B, we may make use of eq. (69), thus :—

$$t^2_{\text{UAR}} = v^2 + \frac{16}{9}\,\frac{\gamma^2}{\pi^2 L^2} = 10^{-9} \times 5{\cdot}289 + 10^{-9} \times 5{\cdot}615 = 10^{-9} \times 10{\cdot}904.$$

Also,

$$t^2_{\text{RAB}} = 10^{-9}(5{\cdot}289 + 2{\cdot}440) = 10^{-9} \times 7{\cdot}729.$$

For the other surface angles the transectors are used, thus—

$$t_{\text{B}}^2 = v^2 + \left(\frac{4}{3}\,\frac{\gamma j}{\pi L_1 L_2}\right)^2 = 10^{-9}(5{\cdot}289 + 17{\cdot}03) = 10^{-9} \times 22{\cdot}32,$$

$$t_{\text{C}}^2 = 10^{-9}(5{\cdot}29 + 11{\cdot}48) = 10^{-9} \times 16{\cdot}77.$$

Now, the average error in the bearing of U A (β'_1) is compounded of that in the bearing of the reference (namely, $\pm$ 15″), and that in the angle U A R (namely, t_{UAR}); therefore

$$\beta'^2_1 = 10^{-9}(5{\cdot}289 + 10{\cdot}904) = 10^{-9} \times 16{\cdot}193.$$

The bearing of A B is also obtained directly from the reference; therefore

$$\beta'^2_2 = 10^{-9}(5{\cdot}289 + 7{\cdot}729) = 10^{-9} \times 13{\cdot}018.$$

Since the bearing of B C is derived from that of A B, we have—

$$\beta'^2_3 = 10^{-9}(13{\cdot}02 + 22{\cdot}32) = 10^{-9} \times 35{\cdot}34,$$

and $$\beta'^2_4 = 10^{-9}(35{\cdot}34 + 16{\cdot}77) = 10^{-9} \times 52{\cdot}11.$$

The underground line E F is permanently marked, and will act as reference for all future underground surveys. It is, therefore, important to find how accurately its bearing is established by this process of shaft-connection. Now, the error in bearing of E F will depend, first, on the accuracy of the surface traverse, and, secondly, on that of the underground traverse. As the underground traverse is oriented from the bearing of U D, as calculated from the surface work, errors in the surface survey will have the power to disturb the bearing of the underground lines by just so much as they are able to affect the bearing of U D.

It is, therefore, necessary to evaluate the average error in the bearing of U D. Roughly its bearing is 74°. If, then, we take an arbitrary meridian running N. 74° E., and, after expressing the courses of the surface lines to this meridian, find the average error in the departure of D, we can quickly convert that result into the average error in the bearing of U D, since the length of that line is known. The method of arriving at this result is identical with that used in the last problem ; the tabular mode of representing it may, therefore, be adopted without detailed explanation :—

Lines.	Lengths, L Feet.	Arbitrary Bearings, β.	β'^2.	Log $\sin^2\beta$.	Log $\cos^2\beta$.	L $\sin^2\beta$.	$L^2\beta'^2\cos^2\beta$.
U A	118	N. 55° W.	$10^{-8}\times 1{\cdot}619$	$\bar{1}{\cdot}8267$	$\bar{1}{\cdot}5172$	79·18	$10^{-4}\times 0{\cdot}742$
A B	179	N. 5° E.	$10^{-8}\times 1{\cdot}302$	$\bar{3}{\cdot}8806$	$\bar{1}{\cdot}9967$	1·36	$10^{-4}\times 4{\cdot}140$
B C	106	N. 17° W.	$10^{-8}\times 3{\cdot}534$	$\bar{2}{\cdot}9319$	$\bar{1}{\cdot}9612$	9·06	$10^{-4}\times 3{\cdot}632$
C D	116	N. 78½° E.	$10^{-8}\times 5{\cdot}211$	$\bar{1}{\cdot}9824$	$\bar{2}{\cdot}5993$	111·40	$10^{-4}\times 0{\cdot}279$
					Totals,	201·00	$10^{-4}\times 8{\cdot}793$

$$K_2^2\,\Sigma\,(L\sin^2\beta) = (0{\cdot}0063)^2 \times 201 = 10^{-3} \times 7{\cdot}977.$$

Square of average error in departure of D *due to surface errors*

$$= \{10^{-3} \times 7{\cdot}977 + 10^{-3} \times 0{\cdot}879\} = 10^{-3} \times 8{\cdot}856.$$

By taking the square root of this result and dividing by the length of U D, we could obtain the average error in the bearing of U D in radians. The form, as it stands at present, is, however, more useful.

The next step is to ascertain the average influence of the errors in the underground traverse. The problem of orienting the underground survey is really that of fitting the extremities of a stiff figure U F E D, one to the fixed point U, and the other to the fixed point D. It follows that, if the figure is mis-shapen through error, the line E F, after the process of fitting, will be moved in azimuth an amount, say, x because of that error. A little consideration will show that if we were to take the direction of E F as fixed, and make use only of that direction in placing the stiff figure, the line U D would be swung by the underground errors through an angle which would also equal x. This indirect mode of finding x is put to service below, β' of line E F being taken as zero for the purpose of the calculation, and the average swing due to underground errors being expressed as a disturbance of the point D. The direction of U D is again used as arbitrary meridian, so as to allow of the application of eq. (79).

Lines.	Lengths, L Feet.	Arbitrary Bearings, β.	β'^2.	Log $\sin^2\beta$.	Log $\cos^2\beta$.	L $\sin^2\beta$.	$L^2\beta'^2\cos^2\beta$.
D E	192	N. 14° W.	$10^{-9}\times 7{\cdot}492$	$\bar{2}{\cdot}7674$	$\bar{1}{\cdot}9738$	11·24	$10^{-4}\times 2{\cdot}60$
E F	183	N. 4° W.	0	$\bar{3}{\cdot}6872$	...	0·89	...
F U	60	N. 89° E.	$10^{-9}\times 11{\cdot}142$	$\bar{1}{\cdot}9999$	$\bar{4}{\cdot}4837$	59·99	Negligible.
					Totals,	72·12	$10^{-4}\times 2{\cdot}60$

$$K_2^2\,\Sigma(L\sin^2\beta) = (0{\cdot}0063)^2\times 72{\cdot}12 = 10^{-3}\times 2{\cdot}862.$$

Square of average error in departure of D *due to underground errors*

$$= 10^{-3}(2{\cdot}862 + 0{\cdot}260) = 10^{-3}\times 3{\cdot}122.$$

This amount, if expressed in angular measure, would represent the average swing of E F due to underground error; also the result obtained above for the effect of surface error on the departure of D, if similarly expressed, would likewise represent the average swing of E F (or any underground line) due to surface error; therefore, the two results must act in combination in affecting the bearing of this line. By adding them and dividing the result by the length of U D we obtain their combined influence, thus—

$$\left.\begin{array}{l}\textit{Average error in the}\\ \textit{bearing of } \mathrm{E\,F}\end{array}\right\} = \frac{\sqrt{10^{-3}(8{\cdot}856+3{\cdot}122)}}{369}\ \text{radians,}$$

or 61 seconds.

We may, therefore, state that there is approximately a 4 to 3 chance of the *actual* error in this bearing being less than 1 minute, a 3 to 1 chance of it being less than $1\frac{1}{2}$ minutes, and an 8 to 1 chance of it being below 2 minutes (*see* Table III, Appendix).

CHAPTER VI.

THE PROPAGATION OF ERROR IN MINOR TRIANGULATION.

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40. **Relative Accuracy of Repetition and Reiteration.**—The two chief methods of measuring angles in triangulation surveys are those most commonly known as *repetition* and *reiteration*.*

To put the matter as briefly as possible, in repetition an angle is multiplied a number of times on the graduated limb, and the required result obtained by dividing the total or multiple angle by the number of repetitions, while in reiteration the angle is ascertained as the mean of a number of simple measurements made on various parts of the graduated circle. To take three repetitions, for example, means that the angle in question is multiplied three times mechanically by the theodolite, and the result obtained by dividing the total movement of each vernier by three, and then finding the mean of the two quotients. By three reiterations is meant that the angle is read three times by each vernier. The first time the left-hand sight is made to read zero on one of the verniers; the second time it is made to read 60° on the same vernier; and the third time 120°, thus ensuring that all parts of the circle are utilised by one or other of the two verniers. By three repetitions on each face one means that the angle is first obtained from three multiplications with the telescope of the transit theodolite in the normal position: then the telescope is reversed, and the angle again measured in

* Most text-books on surveying give descriptions of these methods. *See* Middleton and Chadwick's "Treatise on Surveying," for instance, Part I., p. 191 (Second Edition).

exactly the same way. Similarly, in three reiterations on each face, the angle is measured six times—three face-left and three face-right—and, since there are two verniers, twelve readings of the angle are obtained, and the arithmetic mean taken.

The question now arises as to which of these methods is the better.

It has already been shown, in § 31, that the effects of centring errors are usually small enough to be neglected in triangulation, and that the sighting-and-reading error, v, plays the chief *rôle* in affecting the accuracy of the angles in this class of survey. We shall here make an attempt to divide the error v into its constituent parts.

Let α_r seconds be the average error in taking one reading of a vernier, and α_s seconds be the average error in sighting a station, α_r being held to include the effects of uneven graduation of the plate, and α_s the minor errors due to imperfect levelling and centring, together with the residuum of instrumental errors not perfectly eliminated by the act of measuring the angles on both faces.

Consider an angle measured by n repetitions on each face. First, dealing with errors of reading only, each vernier is read twice in obtaining the multiple angle on each face. The average influence on the multiple angle is, therefore, $\pm \alpha_r \sqrt{2}$ per vernier. When the multiple angle is divided by n, this error will be reduced to $\pm \alpha_r \sqrt{2} \div n$. The fact that both verniers are read and the mean of their measurements taken causes the reading error to be reduced to $\pm \frac{\alpha_r \sqrt{2}}{n} \div \sqrt{2}$ or $\pm \frac{\alpha_r}{n}$ for each face, and since exactly similar readings are taken on the other face, the influence of reading error on the final result is further reduced to $\pm \frac{\alpha_r}{\sqrt{2}\, n}$.

Secondly, dealing with errors of sighting only, $2n$ sights are taken on each face, and their influence on the multiple angle is accordingly $\pm \alpha_s \sqrt{2n}$. When the multiple angle is divided by n the effect of the sighting errors on the quotient is $\pm \alpha_s \sqrt{2n} \div n$ or $\pm \alpha_s \sqrt{\frac{2}{n}}$, and by taking the mean of the quotients as obtained from the two faces, the error in question is reduced to $\pm \alpha_s \sqrt{\frac{2}{n}} \div \sqrt{2}$ or $\pm \frac{\alpha_s}{\sqrt{n}}$.

Combining the errors of reading and sighting, we get—

Average error in an angle measured by n *repetitions on each face*

$$= \pm \sqrt{\left\{ \frac{\alpha_r^2}{2n^2} + \frac{\alpha_s^2}{n} \right\}} \text{ seconds.} \quad . \quad . \quad (84)$$

Now, consider the case of an angle measured by n reiterations on each face. Altogether the mean of $2n$ angles is taken, each angle being obtained separately, and measured by both verniers. By reading one angle by one vernier an average reading error of $\pm \alpha_r \sqrt{2}$ is introduced. By taking the mean of both verniers' readings, this error is reduced to $\pm \alpha_r \sqrt{2} \div \sqrt{2}$, or $\pm \alpha_r$; and when the mean of the $2n$ angles is obtained, the reading errors influence the final result by $\pm \frac{\alpha_r}{\sqrt{2n}}$ seconds on the average. To deal next with the sighting error, since there are two sights per angle, the average error per angle due to sighting is $\pm \alpha_s \sqrt{2}$. The final result is derived as the mean of $2n$ angles; therefore, it is influenced by sighting error to the average extent of $\pm \alpha_s \sqrt{2} \div \sqrt{2n}$ or $\pm \frac{\alpha_s}{\sqrt{n}}$. Combining the errors of reading and sighting, we get—

Average error in an angle measured by n *reiterations on each face*

$$= \pm \sqrt{\left\{\frac{\alpha_r^2}{2n} + \frac{\alpha_s^2}{n}\right\}} . \qquad . \qquad . \quad (85)$$

From these expressions it is seen that, while the sighting error is reduced at the same rate by both methods, the reading error is more rapidly reduced by repetition than by reiteration.

Let us carry the investigation a step further by taking the specific case of a 5-inch transit theodolite having verniers graduated to read to 20 seconds, experience with the instrument in minor triangulation having shown an average error of $\pm$ 7 seconds to result in an angle measured by three reiterations on each face.

Now, with a modern instrument of this kind, it is not difficult to distinguish intervals of 10 seconds by estimation. Hence an actual reading of, say, 22° 53′ 03″ would be booked as 22° 53′ 00″, and one of 22° 53′ 07″ as 22° 53′ 10″. The maximum reading error is, therefore, 5 seconds, and, as the error is capable of assuming any magnitude between zero and 5 seconds with equal likelihood, the average error of reading a vernier is $\pm$ 2·5 seconds. Since one cannot depend, even in the best of instruments, on the circle being graduated with absolute regularity, it is necessary to make an allowance for the fact. If we double the last result on this account, we shall arrive at a result for α_r—namely, $\pm$ 5 seconds—which, if anything, will be a little on the safe side. Substituting that value of α_r in eq. (85) we obtain, when $n = 3$—

$$7 = \pm \sqrt{\left\{\frac{5^2}{2 \times 3} + \frac{\alpha_s^2}{3}\right\}},$$

or $\alpha_s = \pm 11{\cdot}5$ seconds as the average sighting error with the instrument named.

These figures, and equations (84) and (85), allow of a rough comparison being made of the precision attainable by the methods of repetition and reiteration with this instrument, which is termed "Theodolite I." in the table below. A similar comparison is made for "Theodolite II.," whose limb is graduated to single minutes only, and in which the divisions are so coarsely engraved as to render any reliable visual estimation of a fraction of a minute hardly possible. In Theodolite II. the average error in reading a vernier is $\pm$ 15 seconds, and α_r is taken as $\pm$ 18 seconds for it, so as to include the effect of uneven graduation. The sighting error, α_s, is taken as $\pm$ 11·5 seconds for both instruments, since their telescopes, clamps, and tangent screws are about equally good. The results, which are derived from (84) and (85), are stated to the nearest half-second :—

Number of Repetitions or Reiterations on each Face (n).	Average Error in Angle, in Seconds ($\pm$).			
	Theodolite I.		Theodolite II.	
	Repetition.	Reiteration.	Repetition.	Reiteration.
1	12	12	17	17
2	8	8·5	10	12
3	6·5	7	8	10
4	5·5	6	6·5	8·5
5	5	5·5	5·5	7·5

From these figures it may be concluded that, with the more finely divided theodolite, there is no appreciable difference in accuracy between the methods, so that preference should be given to that proving more expeditious in the field—namely, reiteration. The advantage of repetition with Theodolite II., however, is marked, and it is seen that the proportional difference between the errors by the two methods increases as n

increases. Within the limits of the table, n repetitions with this second instrument are about as good as $n + 1$ reiterations; hence the field-method to be recommended to secure a given degree of precision depends on whether the operator finds it quicker to take n repetitions or $n + 1$ reiterations.

The greater the preponderance of α_r over α_s—*i.e.*, the more coarsely an instrument is divided, the more apparent do the advantages of repetition become.

Another valuable conclusion which may be gleaned from the table is the slow rate of diminution of the error with the increase in n. For instance, four reiterations are needed on each face in order to halve the average error affecting a single face-left and face-right measurement of an angle,—that is to say, the error is inversely proportional to the square root of the number of reiterations. With repetition the error diminishes at a slightly more rapid rate.

Equation (85) allows of a question being settled which often troubles surveyors—namely, How closely should a vernier reading be taken? Ought one to endeavour to estimate to five seconds, or is ten near enough? Making use of the criterion of negligibility of § 20, we have that if—

$$\frac{\alpha_r}{\sqrt{2}\,n} \overline{\overline{<}} \frac{\alpha_s}{3\sqrt{n}}, \text{ or } \alpha_r \overline{\overline{<}} \alpha_s \frac{\sqrt{2}}{3},$$

the former term may be neglected. When α_s is $\pm$ 11·5 seconds, α_r will, therefore, be negligible when equal to or less than $\pm$ 5·4 seconds. Now, by estimating to 10 seconds, an average reading error of about $\pm$ 5 seconds is given when uneven graduation is taken into account; hence it is permissible to infer that no tangible increase in accuracy would result were Theodolite I to be read more closely than to ten seconds. It would, for example, be a waste of money to equip that theodolite with reading micrometers.

It is hoped that the reader will now be tempted to make an analysis similar to the above to suit his own instruments.

41. **Propagation of Error in a Chain of Triangles.**— Fig. 21 shows a chain of triangles dependent on the measured base, c. When the triangles come to be solved, a will serve as base for the second triangle, d for the third, and so on.

It was proved in § 24 that the best shape of a triangulation triangle for practical purposes is the equilateral; it was also proved (eq. 37) that the average fractional error, $\left(\frac{a_1}{a}\right)$, in the side, a, of the first triangle, no matter what may be the triangle's shape, is—

$$\frac{a_1}{a} = \pm \sqrt{\left\{ v^2(\cot^2 A + \cot^2 C) + \left(\frac{c_1}{c}\right)^2 \right\}},$$

where v is the average error in any one of the angles, and $\frac{c_1}{c}$ the average fractional error in the base.

Similarly—

$$\frac{d_1}{d} = \pm \sqrt{\left\{ v^2(\cot^2 D + \cot^2 F) + \left(\frac{a_1}{a}\right)^2 \right\}}$$

$$= \pm \sqrt{\left\{ v^2(\cot^2 D + \cot^2 F + \cot^2 A + \cot^2 C) + \left(\frac{c_1}{c}\right)^2 \right\}},$$

and if $\frac{z_1}{z}$ be the average fractional error in a side of the nth triangle—

$$\frac{z_1}{z} = \pm \sqrt{\left\{ v^2 \left(\begin{array}{l} \textit{Sum of the squares of the} \\ \textit{cotangents of all the angles of} \\ \textit{the system which are opposite} \\ \textit{internal sides *} \end{array}\right) + \left(\frac{c_1}{c}\right)^2 \right\}}. \quad (86)$$

* All such lines as a, d, g, etc., which are not bounding lines of the area covered by the system, are spoken of here as internal sides.

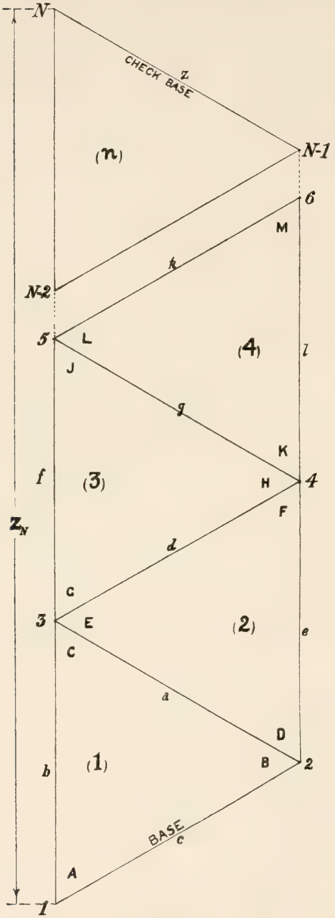

Fig. 21.

In practice, whether the survey is accurate or not is judged from a comparison of the measured and calculated length of a check-base.

If z is this check-base, placed at the extreme end of the scheme, eq. (86) provides a measure of the average fractional error in the *calculated length* of that line.

The *measured length* of the check-base will also be affected by error. Let z_2 be the average error in measuring the line.

Now, the difference between the calculated and measured lengths of the check-base is due to a combination of the error of measurement of the line with that resulting from triangulation imperfections; or, otherwise expressed, the *average* discrepancy (z_3) between the measured and calculated values of z is produced from z_1 and z_2 acting together.

From eq. (15) we have—

$$z_3 = \sqrt{z_1^2 + z_2^2}, \quad . \quad . \quad . \quad (87)$$

or, substituting from eq. (86), and stating the average fractional discrepancy—

$$\frac{z_3}{z} = \pm\sqrt{\left\{ v^2 \begin{pmatrix} \text{Sum of the squares of the} \\ \text{cotangents of all angles} \\ \text{opposite internal sides} \end{pmatrix} + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2 \right\}}. \quad (88)$$

When the triangles are roughly equilateral (88) reduces to—

$$\frac{z_3}{z} = \pm\sqrt{\left\{ \frac{2\,nv^2}{3} + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2 \right\}}. \quad . \quad (89)$$

Although eq. (88) is always safer to use, especially in large systems, this last expression may be employed instead without grave risk even when the triangles depart to some extent from the equilateral shape, for it was shown in § 24 that there is little to choose between isosceles triangles whose apical angles lie anywhere between 50° and 90°.

When a check-base forms a side of the triangle most remote from the base it is generally taken for granted that, if the measured and calculated values for the line agree to, say, 1 in 12,000, the accuracy of the triangulation as a whole may be expressed by that ratio. Later in the chapter some criticism on this assumption is attempted. There can be no doubt, however, that, apart from the safeguard it supplies in detecting grave errors in the field work or calculation, the verification base is of distinct service as affording a measure of precision, and its utility in one important respect falls to be mentioned here :—

A minor triangulation is usually performed to serve as a backbone survey ; by means of it a number of fixed points scattered over the property are established with a high degree of accuracy. These points are afterwards linked together by traverses which are for the purpose of gathering in detail. If the triangulation stations were well placed, there is never need for these traverses to be extensive, the greater part of them running from one triangulation station to the next. Hence —to use the criterion of negligibility of § 20—if the distance between two triangulation stations is known with an average error equal to or less than one-third of that accruing in a traverse connecting them, the closing error of the traverse can be assessed without it being necessary to take the triangulation error into account. For example, if the maximum "accuracy" in one of these traverses is likely to be 1 in 4,000, a suitable accuracy for a triangulation line would be 1 in 12,000.* Should it then be found, on completion of the triangulation, that the two values for the check-base agreed to within one-twelve-thousandth of its length, it could safely be concluded that the required degree of precision had been attained.

When the actual fractional error in a check-base has been determined it is probable that the fractional error in all preceding lines will be less than that amount.

From considerations like those just discussed a surveyor decides on the degree of accuracy required in the lengths of the triangulation lines. The verification base allows him to tell if that precision has been reached.

42. **Connection between the Number of Triangles and the Accuracy of Measurements.**—One of the most important and most difficult questions in triangulation arises before the survey commences and after it has been decided what precision is required, and is, How carefully must the base and angles be measured in order that there may be a fair chance of this degree of accuracy being achieved ?

* It would be more correct to speak of an *accuracy* of 12,000 to 1, and of an *error* of 1 in 12,000 ; yet the term accuracy is so generally applied in the significance given it above, that it seems too late to try to alter matters now.

It ought not to be necessary at this stage to point out that the "fair chance" is all that can be counted on. While it is true, by using extreme care and employing unusual precautions in all measurements, that the odds in favour of a certain accuracy being reached can be made very high, by no methods, however exact, can we ensure the *absolute certainty* of the degree of accuracy in question being attained. Every surveyor has met with the case when, perhaps against all precedent, a bad result has been obtained by good methods and careful work. Such an occurrence ought not to excite surprise, for even if odds of 20 to 1 were to be secured against the final error being outside permissible limits, yet once out of every 21 surveys, on the average, it would exceed them.

By saying, then, that such-and-such methods when applied to a certain triangulation system give an "accuracy" of, say, 1 in 12,000, we mean that, if the survey were to be repeated a large number of times, and the average of the final errors taken, that average, when expressed by the fractional mode, would be of the order stated ; not that those methods would always be sure to give a final error *under* that amount.

A satisfactory answer to the question enunciated above can be obtained by the aid of relation (86), which, when the triangles are approximately equilateral, reduces to the following form :—

$$\frac{z_1}{z} = \pm\sqrt{\left\{\frac{2\,n}{3}v^2 + \left(\frac{c_1}{c}\right)^2\right\}} \quad . \quad . \quad . \quad (90)$$

By substituting what we may roughly term the "accuracy ratio" (1 in 12,000, 1 in 10,000, or whatever it may be) for $\frac{z_1}{z}$ in this equation, suitable values for v and $\frac{c_1}{c}$ may be ascertained. Theoretically the most equitable arrangement is for $\frac{c_1}{c}$ to equal $v\sqrt{\frac{2\,n}{3}}$; but that condition cannot always be satisfied. Circumstances may be such that it is easier to attain a low value of v than a low value of $\frac{c_1}{c}$, in which case $\frac{c_1}{c}$ may profitably be allowed to assume a higher, and v a correspondingly lower, value than those the strictly equitable arrangement would give; for it must be borne in mind that the necessary degree of precision is best secured when the expenditure of labour is a minimum.

Surveyors' methods and instruments differing as they do, dogmatism would be particularly objectionable in dealing with the method of measurement best to apply to attain a certain precision in triangulation. The following table is, therefore, only to be considered as a guide—or, more properly speaking, as an example setting forth the lines on which the reader may construct his own table. It is computed from (90) on the assumption that an "accuracy ratio" of 1 in 12,000 is desired in the check-base—a sufficiently high degree of precision for most minor triangulations—and it suggests suitable values of v and $\frac{c_1}{c}$ for different numbers of roughly equilateral triangles. The mode of measuring the angles entered in the last column refers to the use of a reliable 5-inch transit theodolite, reading to 20″ of arc—such an instrument, indeed, as "Theodolite I." of the table of § 39. It will be noted that the latter table has been utilised in writing the last column.

If, then, the base of a chain of five roughly equilateral triangles were measured with an average error of one-eighteen-thousandth of its length, and the angles with an average error of 7 seconds, the average error in the side most remote from the base would be approximately one-twelve-thousandth of its length. Table III. of the Appendix allows us to assess the odds in favour of gaining this degree of precision when the base and angles are

Number of Triangles in the System. n.	Average Error in Angle in Seconds. $\pm e$.	Average Error in Base. $\pm e_1 \div c$.	Suggested Mode of Measuring Angles.
1	12	1 : 14,500	Single F.L. and F.R. observation.
3	8·5	1 : 17,000	2 reiterations on each face.
5	8·5	1 : 28,000	Ditto.
	or 7	1 : 18,000	3 reiterations on each face.
7	7	1 : 25,500	Ditto.
9	6	1 : 23,500	4 reiterations on each face.

measured with the accuracy stated. The odds are seen to be about 4 to 3.

Now, the author believes most surveyors will agree that odds of 4 to 3 can hardly be taken as a "fair chance," when we consider the trouble and expense of repeating the triangulation in the event of a poor result being obtained. Attention is, however, called to the fact that the values of $\frac{c_1}{c}$ in the third column have purposely been kept above those we expect to secure with base-lines over 500 feet long, measured with a standardised steel tape, when the proper tension is applied and corrections made on account of slope and temperature.*

By aiming at a uniform accuracy ratio of 1 in 50,000 in measuring the bases, the chance of attaining 1 in 12,000 in the check-bases of the systems will be brought higher, in some cases much higher, than 4 to 3. This is shown in the following table, constructed by the aid of eq. (90) and Table III. of the Appendix :—

* See § 8, where the method of obtaining the average error in base measurement from a series of measurements is explained.

Number of Triangles. n.	Average Error in Angle in Seconds. $\pm v$.	Average Error in Base. $\pm \frac{e_1}{c}$.	Odds in favour of an Accuracy Ratio of 1 : 12,000 being obtained in the Check-base.
1	12	1 : 50,000	4 : 1
3	8·5	1 : 50,000	5 : 2
5	{ 8·5	1 : 50,000	3 : 2
	{ 7	1 : 50,000	9 : 4
7	7	1 : 50,000	8 : 5
9	6	1 : 50,000	7 : 4

The difficulty of obtaining heavy odds when n is large—which this table evidences—points to the advisability of arranging the base, if possible, in the centre of a long chain of triangles, rather than at one end. That this leads to better results is widely admitted.

The reader, perhaps, may be inclined to point out that by distributing the error in the triangles by the well-known method derived from the law of least squares, an increase in the accuracy of the angles will be obtained, and that the process will, therefore, increase the odds of column 4. Care must, however, be taken against exaggerating the value of distributing error. It can be shown that the angular error, v, in an angle is only reduced to $v\sqrt{\frac{2}{3}}$, or $0 \cdot 8\, v$, by adjusting the angles of a triangle by means of one "equation of condition." So slight, indeed, is the extra precision attainable in this way, that rather than consider "spreading" the error to increase the odds, it will be safer to regard the process as making a rough compensation for the fact that the triangles in actual systems are never exactly equilateral. This latter fact tends towards reducing the stated odds of the table above, since eq. (90)—which is only strictly true with equilateral triangles—was used when they were calculated.

43. **Example Illustrating the Method of Computing the**

Average Error affecting Triangulation Lines.—It having been decided, after reconnaissance, to lay out a triangulation as is shown in Fig. 22, the angles of the triangles were roughly measured with a prismatic compass, permitting of the figure being drawn. The angles, as given by the compass, are marked on the diagram, and c is the measured base of the system. It is considered that, when the base comes to be measured, an error of about one-fifty-thousandth of its length can be expected on the average.

The scheme is different to the straight chain of Fig. 21 in that the triangles close back about the central point, C This arrangement is likely to give a high degree of accuracy.

It is seen that the side, a, can be calculated in two ways; first, straightway from triangle (1), and, second, by passing round the scheme in a clockwise direction, solving the triangles in the order in which they are numbered, until a is reached as a side of the fifth triangle.

Let us set to work to ascertain what discrepancy will be given on the average between these two values for a, if the angles of the system can each be measured with an average error of $\pm$ 7 seconds.

As some of the triangles depart pretty considerably from the equilateral shape, it will be advisable to make use of the general expression :—

$$\frac{a_1}{a} = \pm \sqrt{\left\{ v^2(\cot^2 A + \cot^2 C) + \left(\frac{c_1}{c}\right)^2 \right\}},$$

and apply it successively to the triangles taken in order, as follows :—

Working round in the clockwise direction, we first need to find the average error in b. To suit this line the above equation can be written—

$$\left(\frac{b_1}{b}\right)^2 = \left\{ v^2(\cot^2 B + \cot^2 C) + \left(\frac{c_1}{c}\right)^2 \right\}$$

$$= \left\{ v^2(\cot^2 91^\circ + \cot^2 50^\circ) + \left(\frac{c_1}{c}\right)^2 \right\}.$$

Now, $v = \pm 7$ seconds, and, in radians, $v^2 = 10^{-9} \times 1{\cdot}152$ (*see* Table II., Appendix). Also $\frac{c_1}{c} = \pm \frac{1}{50{,}000}$, therefore $\left(\frac{c_1}{c}\right)^2 = 10^{-10} \times 4$.

By substituting these values, we find—

$$\left(\frac{b_1}{b}\right)^2 = \{10^{-10} \times 8{\cdot}110 + 10^{-10} \times 4\} = 10^{-10} \times 12{\cdot}11.$$

Taking the second triangle, the side e will be calculated from b as base, and the square of the average fractional error in it will be—

$$\left(\frac{e_1}{e}\right)^2 = \left\{ v^2(\cot^2 48^\circ + \cot^2 50^\circ) + \left(\frac{b_1}{b}\right)^2 \right\}$$
$$= \{10^{-10} \times 17{\cdot}45 + 10^{-10} \times 12{\cdot}11\} = 10^{-10} \times 29{\cdot}56.$$

Treating the other triangles similarly, we obtain—

$$\left(\frac{d_1}{d}\right)^2 = 10^{-10} \times 37{\cdot}42.$$

$$\left(\frac{k_1}{k}\right)^2 = 10^{-10} \times 49{\cdot}31.$$

$$\left(\frac{a_1}{a}\right)^2 = 10^{-10} \times 92{\cdot}74.$$

The effect due to the somewhat ill-conditioned triangle (5) will be noted.

Directly from triangle (1), the average fractional error $\left(\frac{a_2}{a}\right)$ in the side a works out as—

$$\left(\frac{a_2}{a}\right)^2 = \left\{ v^2 (\cot^2 39^\circ + \cot^2 50^\circ) + \left(\frac{c_1}{c}\right)^2 \right\}$$
$$= 10^{-10} \times 29{\cdot}67.$$

Now, the average discrepancy (a_3) between the two values of a will be—

$$a_3 = \pm \sqrt{a_1^2 + a_2^2}.$$

Or, stating it as a fraction of the length of the line—

$$\frac{a_3}{a} = \pm\sqrt{\left(\frac{a_1}{a}\right)^2 + \left(\frac{a_2}{a}\right)^2}$$

$$= \pm\sqrt{10^{-10}(92{\cdot}74 + 29{\cdot}67)} = \pm 10^{-4} \times 1{\cdot}1.$$

Or, $\pm$ 1 in 9,040.

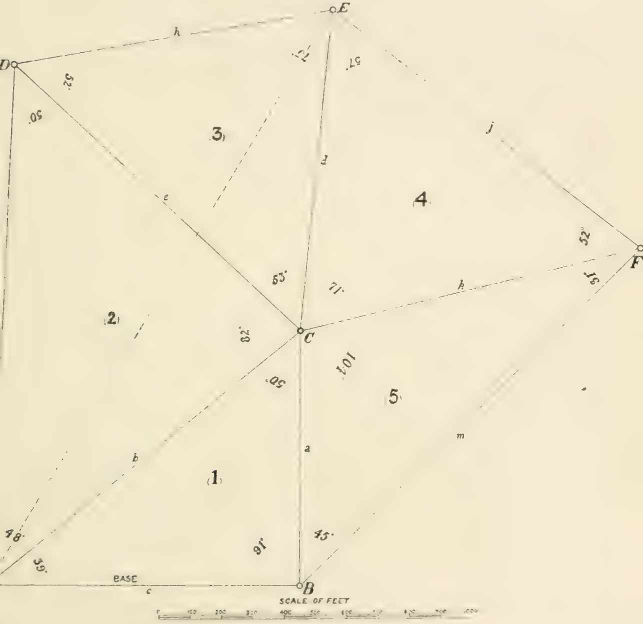

Fig. 22.

All other lines will have an average fractional error less than this, and generally considerably less. There is, therefore, a good chance of the survey being conducted with an accuracy—expressed in terms of the fractional error in any of the lines—greater than 1 in 10,000, and this is all the more true when we remember that, when solving them, the triangles need not be taken altogether in the clockwise order; it will be advisable, indeed, to take triangles (1), (2), and (3) in the clockwise, and (1), (5), and (4) in the anti-clockwise direction.

44. **The Accuracy of Triangulation as a Method of Transmitting Distance.**—Having determined how the various lines of a triangulation scheme are affected by error, it remains to discuss in what way these several errors combine in disturbing the positions of the stations.

Fig. 21 shows a chain of approximately equilateral triangles depending on a measured base, c. Let station 1 be taken as origin of co-ordinates, and line c as arbitrary meridian. An error in any point is stated with regard to station 1 as a fixed point and to the direction of line c as a fixed direction.

In dealing with the average total error affecting, say, the point 7, it is necessary to regard separately the influence of angular and of base error. The effect of the latter is easy to evaluate, since a fractional error in the base will cause the whole system to expand or contract in that proportion; hence, if Z_7 be the distance of the point 7 from the origin, the average fractional error in Z_7, due to base error only, will equal $\pm \frac{c_1}{c}$.

Dealing next with the average displacement of the stations owing purely to angular errors, (90) may be written in the form—

$$z_1 = \pm \sqrt{\left\{ \frac{2\,nv^2c^2}{3} + c_1{}^2 \right\}}, \quad . \quad . \quad (91)$$

since the sides of the triangles are roughly equal; and if the influence of the base error be eliminated from (91) we have—

$$z_1 = \pm\, vc \sqrt{\frac{2n}{3}}, \quad . \quad . \quad . \quad (92)$$

which measures the average error in the length of a side of the *n*th triangle due purely to error in the angles of the scheme.

Now the apex of the first triangle—namely, the station 3—is disturbed by vector errors transmitted along *a* and *b* and of equal amount: hence from (92) and (30), we have—

$$\left.\begin{array}{l}\textit{Average displacement in station}\\ \textit{3 due to angular errors only}\end{array}\right\} = \pm vc\sqrt{\frac{4}{3}}. \qquad (93)$$

This displacement will be transmitted to station 4 *via* line *d*; it will, as it were, gather on its way the error in *d* itself. Station 4 will also be disturbed by error transmitted along *e*, which in this case will merely be e_1, since station 2 is unaffected by angular error. Applying theorem (30) to sum these displacements, we obtain—

$$\left.\begin{array}{l}\textit{Average displacement of station}\\ \textit{4 due to angular error only}\end{array}\right\} = \pm vc\sqrt{\frac{4}{3}+\frac{4}{3}+\frac{4}{3}}$$

$$= \pm v\sqrt{\frac{12}{3}}. \qquad (94)$$

Similarly the error influencing station 5 is compounded of those influencing stations 3 and 4, together with the errors in the lines *f* and *g*, and may be expressed—

$$\left.\begin{array}{l}\textit{Average displacement of station}\\ \textit{5 due to angular error only}\end{array}\right\} = \pm vc\sqrt{\frac{28}{3}}. \qquad (95)$$

In like manner the average displacements in the remaining stations due purely to angular error may be calculated: they are as follows:—

$$\left.\begin{array}{l}\textit{Average effect of angular}\\ \textit{error on station 6}\end{array}\right\} = \pm\sqrt{\frac{56}{3}}. \qquad (96)$$

$$\text{..} \qquad \text{..} \qquad 7 = \pm\sqrt{\frac{104}{3}}. \qquad (97)$$

Average effect of angular error on station N

$$= \pm \sqrt{\left\{\begin{array}{c}\text{Sum of squares of coefficients of } vc \text{ for stations} \\ (\text{N} - 1) \text{ and } (\text{N} - 2) \textit{ plus } \frac{4}{3}(\text{N} - 2)\end{array}\right\}}. \quad (98)$$

If Z_N be the distance of station N from the origin, the average *fractional error* in that distance, due to angular error in the scheme, is—

$$\frac{Z'_N}{Z_N} = \pm \frac{vc}{Z_N} \sqrt{\left\{\begin{array}{c}\text{Sum of squares of coefficients of } vc \text{ for stations} \\ (\text{N} - 1) \text{ and } (\text{N} - 2) \textit{ plus } \frac{4}{3}(\text{N} - 2)\end{array}\right\}}. \quad (99)$$

We have said already (p. 138) that the average fractional error in Z_N due to an error, c_1, in base measurement is $\pm \frac{c_1}{c}$, and we are now in a position to compound these errors to obtain—

Average fractional error in Z_N *due to linear and angular errors combined*

$$= \pm \sqrt{\left\{\left(\frac{Z'_N}{Z_N}\right)^2 + \left(\frac{c_1}{c}\right)^2\right\}}. \quad . \quad . \quad (100)$$

In this result it will be noticed that, while the effect of the angular error increases as the number of triangles increases, the influence of the fractional error in base measurement is constant ; hence where greater precision of distance-transference is desired in an extensive scheme, it will generally pay better to devote attention to improving the angular rather than the linear measurements.

Another fact of the first importance established by the last result is the necessity of using as *few* triangles as possible in the scheme to cover the area under survey. As accuracy in distance-transmission is desired of every survey, this conclusion is hardly second in importance to the one that the triangles should be well-conditioned.

45. **The Value of the Check-base as a Means to Assessing the Error in Distance Transmission.**—The value of the check-base in allowing of the error in the *sides* of the

triangles being roughly assessed has already been shown; we have yet to examine whether the wider claim—that the accuracy of the triangulation *as a whole* is determined by the discordance (expressed as a ratio) between the measured and calculated length of the line—is justifiable.

If the average fractional error in the check-base were exactly equal to the fractional error in Z_N (the distance of station N from station 1), it would follow that the right-hand sides of equations (89) and (100) would be equal, and it is evident that this is not generally the case. Since, then, the above-stated claim cannot be established in full, it remains to determine between what limits it is admissible.

The table below is drawn up to compare the average fractional error in Z_N with the average fractional difference between the calculated and measured values of the check-base, z, for the scheme of Fig. 21. It is assumed that the check-base forms a side of the triangle whose apex is station N.

The condition to be satisfied in order that the fractional error in the position of, say, the apex of the ninth triangle shall be equal to the proportional difference between the measured and calculated check-base is that $5{\cdot}79\, v^2$ shall be equal to $\left(\frac{z_2}{z}\right)$; or, if v were 6 seconds ($10^{-5} \times 2{\cdot}91$ radians), $\frac{z_1}{z}$ would have to be about 1 in 14,250—a degree of accuracy inferior to that usually attained in practice in the measurement of a check-base for a scheme of this size. We may, therefore, conclude that if the triangles were arranged as in fig. 21, it would not be safe to consider the actual degree of agreement between the two values for the length of the check-base as a measure of the accuracy in position of the apex of the ninth triangle. Again, fig. 21 shows an exceptionally favourable arrangement of triangles for

Station N.	Average Fractional Error in Z_N. (X).	Average Fractional Difference between Measured and Calculated Check-base. (Y).	$X^2 - Y^2$.
Apex of 3rd triangle	$\pm\sqrt{\left\{2{\cdot}33\, v^2 + \left(\frac{c_1}{c}\right)^2\right\}}$	$\pm\sqrt{\left\{2\, v^2 + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2\right\}}$	$0{\cdot}33\, v^2 - \left(\frac{z_2}{z}\right)^2$
,, 5th ,,	$\pm\sqrt{\left\{3{\cdot}85\, v^2 + \left(\frac{c_1}{c}\right)^2\right\}}$	$\pm\sqrt{\left\{3{\cdot}33\, v^2 + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2\right\}}$	$0{\cdot}52\, v^2 - \left(\frac{z_2}{z}\right)^2$
,, 7th ,,	$\pm\sqrt{\left\{6{\cdot}58\, v^2 + \left(\frac{c_1}{c}\right)^2\right\}}$	$\pm\sqrt{\left\{4{\cdot}67\, v^2 + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2\right\}}$	$1{\cdot}91\, v^2 - \left(\frac{z_2}{z}\right)^2$
,, 9th ,,	$\pm\sqrt{\left\{11{\cdot}79\, v^2 + \left(\frac{c_1}{c}\right)^2\right\}}$	$\pm\sqrt{\left\{6\, v^2 + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2\right\}}$	$5{\cdot}79\, v^2 - \left(\frac{z_2}{z}\right)^2$
,, 11th ,,	$\pm\sqrt{\left\{22{\cdot}08\, v^2 + \left(\frac{c_1}{c}\right)^2\right\}}$	$\pm\sqrt{\left\{7{\cdot}33\, v^2 + \left(\frac{c_1}{c}\right)^2 + \left(\frac{z_2}{z}\right)^2\right\}}$	$14{\cdot}75\, v^2 - \left(\frac{z_2}{z}\right)^2$

the rapid transference of distance; almost every other arrangement of the same triangles would result in a diminished Z_N, and, therefore, in an increased fractional error in Z_N. So far, then, this investigation has shown that the average fractional error in the check-base is always less than, and in many cases will be much less than, the average fractional error in distance-transmission when the number of triangles is nine. On the other hand, it can be shown in like manner that if the scheme were composed of only two or three triangles the accuracy in distance-transmission would be superior, on the average, to that in the check-base. There must, therefore, be a value of n, the number of triangles, at which the average fractional error in each of these quantities is about equal, and, to be on the safe side, the writer places this at $n = 5$.

Thus we are led to conclude that the fractional error in the verification base, as determined by comparing its measured and calculated length, may be taken, in general, as a reliable check on the accuracy of distance-transmission only if the number of triangles is less than five; but that when the triangles exceed five in number the check ceases to have value, *owing to the rate of increase of the error in* Z_N *being more rapid than that in the check-base.*

In this connection lies the chief utility of this portion of the investigation, for it provides a criterion on the accuracy of distance-transmission when the check-base ceases to afford a safe measure of it.

46. **Example Illustrating the Method of Evaluating the Accuracy of Distance-Transmission in an Actual Triangulation.**—The results derived in the last two sections are based on the assumption that the system in question consists of equilateral triangles, and while the conclusions are sufficiently near the truth when the triangles depart some little way from the equilateral form, they cannot be expected to hold for systems containing ill-

conditioned triangles, or even for those including triangles approaching the permissible deviation from the best shape. Fig. 22 shows a system in which some of the angles are considerably greater, and others considerably less, than 60°.

Without using any result of §§ 43 and 44, we now intend to determine the average error in position of the point E, with reference to A as a fixed point and to A B as a fixed direction. The figure, it will be remembered, is a protractor-draft from a reconnaissance survey. It is intended to use a method of measuring the angles with the theodolite such as will be likely to give an average error of ± 7 seconds in each, and the base, c, will be obtained by the steel band with an average error of about 1/50,000th of its length. A rough measurement of the base, made by pacing it two or three times, showed its length to be about 1,000 feet. The other lengths are obtainable with sufficient exactitude for the purpose of this analysis by scaling the drawing ; they are stated in the columns below.

Leaving the error in base-measurement out of consideration until near the end of the process, the first step consists of finding the average error in each of the sides due purely to error in the angular measurements.

Now, it is seen that the position of E could be ascertained, when the field work is completed, either by means of the triangles (1), (2), and (3), or by means of the triangles (1), (5), and (4). It is instructive to work out the average disturbance in the position of E affecting *both* those calculations, in order to see which result would be most worthy of trust.

Taking triangles (1), (2), and (3) *in order.*—The average fractional error in a, *due to angular error only*, is given by—

$$\left(\frac{a_1}{a}\right)^2 = v^2 (\cot^2 A + \cot^2 C)$$

from eq. (37). Converting $v = \pm 7$ seconds into radians, and substituting A = 39°, C = 50° in this expression, we get—

$$\left(\frac{a_1}{a}\right)^2 = 10^{-10} \times 25{\cdot}67.$$

Likewise—

$$\left(\frac{b_1}{b}\right)^2 = 10^{-10} \times 8{\cdot}11.$$

Now b serves as base for triangle (2); therefore, by (37)—

$$\left(\frac{e_1}{e}\right)^2 = v^2(\cot^2 48^\circ + \cot^2 50^\circ) + \left(\frac{b_1}{b}\right)^2$$

$$= 10^{-10} \times 25{\cdot}56.$$

and
$$\left(\frac{g_1}{g}\right)^2 = 10^{-10} \times 16{\cdot}45.$$

The results for $\left(\frac{h_1}{h}\right)^2$ and $\left(\frac{d_1}{d}\right)^2$ are obtained similarly. All these results are entered in the second column of the table below.

By multiplying each of the results by the square of the length of the corresponding line, the squares of the average errors in the various lines are derivable; they are given in the fourth column.

Error Due Purely to Angular Error.			
Lines.	Squares of Fractional Errors. $\left(\frac{x_1}{x}\right)^2$.	Lengths of Lines from Diagram. x Feet.	Squares of Average Errors in Lines. x_1^2.
c	Zero.	1,000	Zero.
a	$10^{-10} \times 25{\cdot}67$	815	0·00170
b	$10^{-10} \times 8{\cdot}11$	1,290	0·00107
g	$10^{-10} \times 16{\cdot}45$	1,670	0·00459
e	$10^{-10} \times 25{\cdot}56$	1,255	0·00403
h	$10^{-10} \times 32{\cdot}93$	1,035	0·00353
d	$10^{-10} \times 33{\cdot}42$	1,030	0·00355

Making use of eq. (30), we can now write the average disturbance in position of each of the stations of the left-hand side of the figure, due purely to angular error—

Error in position of A, .	zero.
Error in position of B, .	zero.
Error in position of C, .	$\sqrt{a_1^2+b_1^2}$.
Error in position of D, .	$\sqrt{a_1^2+b_1^2+g_1^2+e_1^2}$.
Error in position of E, .	$\sqrt{2(a_1^2+b_1^2)+g_1^2+e_1^2+h_1^2+d_1^2}$

Or, using the values in the last column of the table—

$$\left.\begin{matrix}\textit{Average error in position of}\\ \text{E } \textit{due to angular error}\end{matrix}\right\} = \sqrt{0{\cdot}02124} = 0{\cdot}146 \text{ foot.}$$

Now, the distance of E from A is 2,150 feet ; hence the average fractional error in that distance will be—

$$\left.\begin{matrix}\textit{Average fractional error in}\\ \text{A E } \textit{due to angular error}\end{matrix}\right\} = \frac{0{\cdot}146}{2{,}150} \text{ or } \frac{1}{14{,}750}.$$

The effect of error in base measurement is to increase or diminish any line of the system by the same proportional amount—namely, $\frac{c_1}{c}$. In this case $\frac{c_1}{c} = 10^{-5} \times 2$. Combining the effects of angular and base-line errors, we obtain—

$$\left.\begin{matrix}\textit{Average fractional}\\ \textit{error in } \text{A E}\end{matrix}\right\} = \sqrt{\left(\frac{1}{14{,}750}\right)^2 + \left(\frac{1}{50{,}000}\right)^2}.$$

In this result the influence of the second term under the radical is so slight that we may write—

$$\left.\begin{matrix}\textit{Average fractional}\\ \textit{error in } \text{A E}\end{matrix}\right\} = \frac{1}{14{,}750}. \qquad . \quad . \quad . \quad (101)$$

Taking Triangles (1), (5), *and* (4) *in Order.*—Working round the right-hand side of the system in exactly the same manner, we derive the results given in the following table :—

ERROR DUE PURELY TO ANGULAR ERROR.			
Lines.	Squares of Fractional Errors. $\left(\frac{x_1}{x}\right)^2$.	Lengths of Lines from Diagram. x Feet.	Squares of Average Errors in Lines. x_1^2.
c	Zero.	1,000	Zero.
a	$10^{-10} \times 25{\cdot}67$	815	0·00170
b	$10^{-10} \times 8{\cdot}11$	1,290	0·00107
m	$10^{-10} \times 58{\cdot}29$	1,530	0·01364
k	$10^{-10} \times 69{\cdot}10$	1,110	0·00851
j	$10^{-10} \times 75{\cdot}32$	1,240	0·01158
d	$10^{-10} \times 80{\cdot}99$	1,030	0·00859

As before, we obtain by the aid of eq. (30)—

$$\left.\begin{array}{r}\textit{Average error in position}\\ \textit{of E due to angular error}\end{array}\right\} = \sqrt{2(a_1^2+b_1^2)+m_1^2+k_1^2+j_1^2+d_1^2}$$

$$= \sqrt{0{\cdot}0478} = 0{\cdot}219 \text{ foot.}$$

As compared with this, the effect of base-line error is negligible ; hence—

$$\left.\begin{array}{r}\textit{Average fractional error}\\ \textit{in A E}\end{array}\right\} = \frac{0{\cdot}219}{2{,}150} = \frac{1}{9{,}830}. \qquad (102)$$

A comparison of (101) and (102) shows that the position of E will be more accurately fixed by the left-hand triangles than by the others. Indeed, as the weight is inversely proportional to the square of the average error (§ 9), we may say that the weight of the two determinations of the position of E will be as 2·25 to 1, respectively. The lower accuracy of the result as obtained from triangles (1), (5), and (4) is due to the poor shape of triangle (5), and also to the fact that triangles (5) and (4) depend on the short line, B C.

CHAPTER VII.

SUMMARY OF RESULTS.

CHAPTER VII.

SUMMARY OF RESULTS.

THE most important of the conclusions reached in Chapters III. to VI. inclusive are here collected in an abridged form. In order to differentiate between new results and those already well known to surveyors, the latter are indicated by asterisks. By so marking them, attention is called to the number of the results deduced which conform with general experience.

In Chapter II. the best shape of triangles is discussed, and the following conclusions are reached :—

When three sides of a triangle are given, to find the area, the best shape is a highly attenuated isosceles triangle, though the equilateral is the most economical shape so far as its area-capacity is concerned.

When two sides and the included angle are given, to find the area, the best and most economical shape is that in which the measured angle is 90°, and the measured sides are equal (§ 23). Greater precision is likely to be obtained when those sides are long than when they are short.

When three angles and one side are given, to calculate the other sides (the triangulation case), the best shape is an isosceles triangle having the apical angle 67° 30′. Out of considerations given, however (§ 24), it is concluded that the equilateral is the most suitable shape for practical purposes.* Curves are drawn (Fig. 7) to show that there is little appreciable difference between the equilateral and a triangle of the theoretical best shape.

When the base is measured with a similar degree of precision to the angles, or more precisely than them, no angle as a main triangle ought to be over 120° or under 30°.* But should the accuracy of angular measure be considerably greater than that of the base, these limits may be set further apart with advantage.

A consideration of §§ 43 and 45 shows that to measure the angles more carefully than the base is not so illogical a proceeding as might at first be supposed, especially if the triangulation system is formed of a large number of triangles.

It is permissible to use triangles towards the end of the system, departing further from the perfect shape than could be allowed in those near the beginning.*

Too great a stress can be laid on considerations of mere shape of triangulation triangles; even when the base and angles are measured with the same order of precision, isosceles triangles in which the apical angles lie between 50° and 90° are all almost equally well-conditioned (§ 24).

When three sides and one angle are given, to calculate the remaining angles (as in the Weisbach mode of shaft-connection), the best shape is that in which the measured angle approaches zero very closely, and the triangle is as thin as possible * (§ 25). Taking the side opposite the measured angle as constant, better results will be obtained when the other sides are short than when they are lengthy.†

In general practice, when the angle at the instrument is less than 25 minutes, the effects of errors in measuring the sides are so small that there is no need to measure them with more than ordinary care.

When the measured angle is less than 25 minutes, the error in either of the calculated angles is directly proportional to that in the measured one (eq. 45); hence

† Generally admitted by surveyors, but not unanimously.

the angle at the instrument should be obtained with as great a precision as possible. If the measured angle exceeds 5°, less attention needs to be paid to that angle, and much more to getting the precise lengths of the sides.

When two sides and the included angle are given to calculate the remaining side (as in a "broken" base-line), the best shape is that in which the two known sides are equal and the known angle very nearly 180° * (§ 26).

Providing that the measured angle lies between 165° and 180°, and the known sides are roughly equal in length, the effect of angular error is inappreciable as compared with that of linear errors ; hence the angle need not be measured with more care than is taken with any other triangulation angle.

The longer the known sides the more precise the result for the third side.*

When three sides are given, to calculate one angle, the theoretical best shape is that having the required angle very nearly zero, and the two sides enclosing it equal in length. It is shown that this shape, however, is not practicable in two methods which are discussed in § 27 to exemplify this case, a fact which militates against the accuracy of those methods. Results are derived for the shapes of triangles of this type which better conform to practical conditions without causing heavy error to be introduced in the calculated angle.

Chapter IV. is concerned with the study of error in traverse surveys, and many of the results derived are applied in Chapter V. to actual cases taken from practice.

A method of evaluating the average error due to imperfect centring of a surveying instrument is evolved (§ 29), allowing of this class of angular error being studied in relation to that due to imperfect sighting and reading of the instrument.

The average angular error due to eccentric centring depends on the value of the traverse angle, being a maximum when that angle is 180°. Hence open traverses are more prone to error than closed traverses, apart from all consideration of the slight additional accuracy obtainable in the latter type from a distribution of error in the angles and co-ordinates.

Displacements in centre have a greater influence on short than on long lines; hence the necessity for closer centring on short lines.*

Centring errors in triangulation are generally negligible.* †

The average effect of eccentric centring on a traverse angle is directly proportional to the distance between the fore and back stations, and inversely to the product of the lengths of the two traverse lines in question (eq. 66).

The average displacement in centre is two-thirds of the maximum permissible displacement (eq. 65).

In compass traversing (loose-needle work) greater accuracy is attainable with short lines, and in theodolite traversing by using long lines.*

The fact that a *superior* accuracy may be attainable by a compass than by a theodolite when the lines are very short is admitted by some.‡ A proof of this is given (§ 38), and it is shown that the length of line for which the accuracy becomes equal for the two kinds of instrument is capable of being roughly calculated for any case arising in practice. The length in question increases as the total length of the traverse increases.

† See Middleton and Chadwick's *Treatise on Surveying*, 1904, Part I., p. 193, for instance.

‡ For example, see "Notes on Railway Surveying," by C. J. Albrecht, *Min. Proc. Inst. C.E.*, vol. cliv., p. 262, where a compass instrument is advised for running trial lines in country overgrown by forest, and over broken ground.

Methods are evolved (§§ 36, 37), and illustrated by actual instances (§ 39), which allow of the average total error being computed in any traverse, whether closed or open, and whether run by theodolite or compass instrument. By their application it is possible to determine, for example, whether the actual error of closure of a polygon is greater or less than that which might have been expected on the average, and thus they supply a criterion of accuracy more satisfactory and informative than that afforded by expressing the closing error as a fraction of the total length of the traverse. Providing a rough draft of the proposed traverse can be had (say from an old plan), these methods permit of information being obtained beforehand on the displacement *which might reasonably be expected* of any point of the traverse ; they also allow one to find approximately the displacement likely to occur in any given direction for any point, and, further, render it possible to assess the degree of precision likely to be attained in any result calculated from data to be derived from the traverse.

The tedium of such computations is not so much as would at first appear : the use of transectors or reciprocal transectors (§ 30) greatly reduces the time required in obtaining the average error due to eccentric centring, and the tables of the Appendix facilitate other parts of the work.

The most important results attained in Chapter VI., which deals with the effects of error in triangulation, are as follows :—

In § 40 some conclusions are advanced as to the relative accuracy of the two methods of measuring angles, known commonly as " repetition " and " reiteration."

In a triangulation system covering a certain area of ground, the fewer the triangles the better * (§ 44). The importance of this fact is emphasised.

When the number of triangles in a trigonometrical survey is large, it is advisable to arrange the base in the centre of the system, rather than at one end.*

Having regard to the relative accuracy in base and angles usually attained in practice, it is generally more desirable to increase the accuracy of the angular rather than of the linear measurements when a greater precision in distance-transmission is desired of a trigonometrical survey (§ 44).

After comparing the measured and calculated length of a verification base, one is able to infer that it is probable that the individual lines of the survey lying between the base and check-base are determined more accurately than is indicated by the ratio derived from the comparison.* For example, if the difference between the two values for the length of the verification base is, say, one-twelve-thousandth of that length, it is permissible to consider that the length of another triangle side not so remote from the base is affected by a fractional error less than that amount.

Usually it is assumed that the actual fractional error in the check-base can be held as representative of the accuracy of the triangulation as a whole. This is true only in part. It is, indeed, dangerous to consider that fraction to express even approximately the accuracy in distance-transmission unless the number of triangles is less than five (§ 45).

The error in distance-transmission increases at a more rapid rate than the error in the individual lines, especially when the triangles are numerous.

A method is developed (§§ 42, 43, 46), which allows of the average error in the length of any line, and that in the position of any station of actual triangulations to be ascertained. By these means it is easy to find the relation between the error which may reasonably be expected in the check-base and that to be expected in distance-transmission. Such an analysis, made after reconnaissance, but prior to the actual survey, not only gives valuable aid in settling by what methods to measure the angles and bases; it enables a rough measure being arrived at, when the survey is finished, of the actual error in distance-transmission—for if it had been found that the average fractional error in distance-transmission was likely to be, say, 1·5 times that of the check-base, and if the "accuracy ratio" of the latter actually came out as 1 : 12,000, the actual "accuracy ratio" of distance-transmission could be taken as somewhere about 1·5 times that ratio, or 1 : 8,000. Short of tying each end of the system to points already located with reference to each other by some more accurate trigonometrical survey—the Ordnance Survey, for instance—there seems no other way of getting trustworthy information as to the accuracy in distance-transmission of an actual triangulation.

APPENDIX.

A SET OF TABLES DESIGNED TO FACILITATE THE CALCULATION OF THE AVERAGE ERROR IN SURVEYS.

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A SET OF TABLES DESIGNED TO FACILITATE THE CALCULATION OF THE AVERAGE ERROR IN SURVEYS.

TABLE I.—Square Roots, Squares, Reciprocals, and Squares of Reciprocals of Numbers.

Number, L.	Square-root, $\sqrt{L}$.	Square, L^2.	Reciprocal, $1/L$.	Square of Reciprocal, $1/L^2$.
1	1·00	1	1	1
2	1·41	4	$10^{-3} \times 500$	$10^{-3} \times 250$
3	1·73	9	$\times 333$	$\times 111$
4	2·00	16	$\times 250$	$10^{-4} \times 625$
5	2·24	25	$\times 200$	$\times 400$
6	2·45	36	$\times 167$	$\times 278$
7	2·65	49	$10^{-3} \times 143$	$\times 204$
8	2·83	64	$\times 125$	$\times 156$
9	3·00	81	$\times 111$	$\times 123$
10	3·16	100	$\times 100$	$\times 100$
11	3·32	121	$10^{-4} \times 909$	$10^{-5} \times 826$
12	3·46	144	$\times 833$	$\times 694$
13	3·61	169	$\times 769$	$\times 592$
14	3·74	196	$\times 714$	$\times 510$
15	3·87	225	$\times 667$	$\times 444$
16	4·00	256	$10^{-4} \times 625$	$10^{-5} \times 391$
17	4·12	289	$\times 588$	$\times 346$
18	4·24	324	$\times 556$	$\times 309$
19	4·36	361	$\times 526$	$\times 277$
20	4·47	400	$\times 500$	$\times 250$
21	4·58	441	$10^{-4} \times 476$	$10^{-5} \times 227$
22	4·69	484	$\times 455$	$\times 207$
23	4·80	529	$\times 435$	$\times 189$
24	4·90	576	$\times 417$	$\times 174$
25	5·00	625	$\times 400$	$\times 160$
26	5·10	676	$10^{-4} \times 385$	$10^{-5} \times 148$
27	5·20	729	$\times 370$	$\times 137$
28	5·29	784	$\times 357$	$\times 127$
29	5·39	841	$\times 345$	$\times 119$

TABLE I.—(*Continued*).

Number, L.	Square-root, $\sqrt{L}$.	Square, L^2.	Reciprocal, 1/L.	Square of Reciprocal, $1/L^2$.
30	5·48	900	$10^{-4} \times 333$	$10^{-5} \times 111$
31	5·57	961	$\times 323$	$\times 104$
32	5·66	1,024	$\times 312$	$10^{-6} \times 977$
33	5·74	1,089	$\times 303$	$\times 918$
34	5·83	1,156	$\times 294$	$\times 865$
35	5·92	1,225	$10^{-4} \times 286$	$10^{-6} \times 818$
36	6·00	1,296	$\times 278$	$\times 773$
37	6·08	1,369	$\times 270$	$\times 730$
38	6·16	1,440	$\times 263$	$\times 692$
39	6·24	1,521	$\times 256$	$\times 657$
40	6·32	1,600	$10^{-4} \times 250$	$10^{-6} \times 625$
41	6·40	1,681	$\times 244$	$\times 595$
42	6·48	1,764	$\times 238$	$\times 567$
43	6·56	1,849	$\times 233$	$\times 541$
44	6·63	1,936	$\times 227$	$\times 516$
45	6·71	2,025	$10^{-4} \times 222$	$10^{-6} \times 494$
46	6·78	2,116	$\times 217$	$\times 473$
47	6·86	2,209	$\times 213$	$\times 453$
48	6·93	2,304	$\times 208$	$\times 434$
49	7·00	2,401	$\times 204$	$\times 416$
50	7·07	2,500	$10^{-4} \times 200$	$10^{-6} \times 400$
51	7·14	2,601	$\times 196$	$\times 384$
52	7·21	2,704	$\times 192$	$\times 370$
53	7·28	2,809	$\times 189$	$\times 356$
54	7·35	2,916	$\times 185$	$\times 343$
55	7·42	3,025	$10^{-4} \times 182$	$10^{-6} \times 331$
56	7·48	3,136	$\times 179$	$\times 319$
57	7·55	3,249	$\times 175$	$\times 308$
58	7·62	3,364	$\times 172$	$\times 297$
59	7·68	3,481	$\times 169$	$\times 287$
60	7·75	3,600	$10^{-4} \times 167$	$10^{-6} \times 278$
61	7·81	3,721	$\times 164$	$\times 269$
62	7·87	3,844	$\times 161$	$\times 260$
63	7·94	3,969	$\times 159$	$\times 252$
64	8·00	4,096	$\times 156$	$\times 244$
65	8·06	4,225	$10^{-4} \times 154$	$10^{-6} \times 237$
66	8·12	4,356	$\times 152$	$\times 230$
67	8·19	4,489	$\times 149$	$\times 223$
68	8·25	4,624	$\times 147$	$\times 216$
69	8·31	4,761	$\times 145$	$\times 210$
70	8·37	4,900	$10^{-4} \times 143$	$10^{-6} \times 204$
71	8·43	5,041	$\times 141$	$\times 198$
72	8·49	5,184	$\times 139$	$\times 193$
73	8·54	5,329	$\times 137$	$\times 188$
74	8·60	5,476	$\times 135$	$\times 183$

TABLE I.—(*Continued*).

Number, L.	Square-root, $\sqrt{L}$.	Square, L^2.	Reciprocal, 1/L.	Square of Reciprocal, $1/L^2$.
75	8·66	5,625	$10^{-4}\times 133$	$10^{-6}\times 178$
76	8·72	5,776	× 132	× 173
77	8·78	5,929	× 130	× 169
78	8·83	6,084	× 128	× 164
79	8·89	6,241	× 127	× 160
80	8·94	6,400	$10^{-4}\times 125$	$10^{-6}\times 156$
81	9·00	6,561	× 123	× 152
82	9·06	6,724	× 122	× 149
83	9·11	6,889	× 120	× 145
84	9·17	7,056	× 119	× 142
85	9·22	7,225	$10^{-4}\times 118$	$10^{-6}\times 138$
86	9·27	7,396	× 116	× 135
87	9·33	7,569	× 115	× 132
88	9·38	7,744	× 114	× 129
89	9·43	7,921	× 112	× 126
90	9·49	8,100	$10^{-4}\times 111$	$10^{-6}\times 124$
91	9·54	8,281	× 110	× 121
92	9·59	8,464	× 109	× 118
93	9·64	8,649	× 108	× 116
94	9·70	8,836	× 106	× 113
95	9·75	9,025	$10^{-4}\times 105$	$10^{-6}\times 111$
96	9·80	9,216	× 104	× 108
97	9·85	9,409	× 103	× 106
98	9·90	9,604	× 102	× 104
99	9·95	9,801	× 101	× 102
100	10·00	10,000	$10^{-4}\times 100$	$10^{-6}\times 100$
105	10·25	11,025	$10^{-5}\times 952$	$10^{-7}\times 907$
110	10·49	12,100	× 909	× 826
115	10·72	13,225	× 870	× 756
120	10·95	14,400	× 833	× 694
125	11·18	15,625	× 800	× 640
130	11·40	16,900	$10^{-5}\times 769$	$10^{-7}\times 592$
135	11·62	18,225	× 741	× 549
140	11·83	19,600	× 714	× 510
145	12·04	21,025	× 690	× 476
150	12·25	22,500	× 667	× 444
155	12·45	24,025	$10^{-5}\times 645$	$10^{-7}\times 416$
160	12·65	25,600	× 625	× 391
165	12·85	27,225	× 606	× 367
170	13·04	28,900	× 588	× 346
175	13·23	30,625	× 572	× 327
180	13·42	32,400	$10^{-5}\times 556$	$10^{-7}\times 309$
185	13·60	34,225	× 540	× 292
190	13·78	36,100	× 526	× 277
195	13·96	38,025	× 513	× 263

TABLE I —(*Continued*).

Number, L.	Square-root, $\sqrt{L}$.	Square, L^2.	Reciprocal, 1/L.	Square of Reciprocal, $1/L^2$.
200	14·14	40,000	$10^{-5} \times 500$	$10^{-7} \times 250$
205	14·32	42,025	$\times 488$	$\times 238$
210	14·49	44,100	$\times 476$	$\times 227$
215	14·66	46,225	$\times 465$	$\times 216$
220	14·83	48,400	$\times 455$	$\times 207$
225	15·00	50,625	$10^{-5} \times 444$	$10^{-7} \times 198$
230	15·17	52,900	$\times 435$	$\times 189$
235	15·33	55,225	$\times 426$	$\times 181$
240	15·49	57,600	$\times 417$	$\times 174$
245	15·65	60,025	$\times 408$	$\times 167$
250	15·81	62,500	$10^{-5} \times 400$	$10^{-7} \times 160$
255	15·97	65,025	$\times 392$	$\times 154$
260	16·12	67,600	$\times 385$	$\times 148$
265	16·28	70,225	$\times 377$	$\times 142$
270	16·43	72,900	$\times 370$	$\times 137$
275	16·58	75,625	$10^{-5} \times 364$	$10^{-7} \times 132$
280	16·73	78,400	$\times 357$	$\times 127$
285	16·88	81,225	$\times 351$	$\times 123$
290	17·03	84,100	$\times 345$	$\times 119$
295	17·18	87,025	$\times 339$	$\times 115$
300	17·32	90,000	$10^{-5} \times 333$	$10^{-7} \times 111$
305	17·46	93,025	$\times 328$	$\times 107$
310	17·61	96,100	$\times 323$	$\times 104$
315	17·75	99,225	$\times 318$	$\times 101$
320	17·89	102,400	$\times 312$	$10^{-8} \times 977$
325	18·03	105,625	$10^{-5} \times 308$	$\times 947$
330	18·17	108,900	$\times 303$	$\times 918$
335	18·30	112,225	$\times 298$	$\times 891$
340	18·44	115,600	$\times 294$	$\times 865$
345	18·57	119,025	$\times 290$	$\times 841$
350	18·71	122,500	$10^{-5} \times 286$	$10^{-8} \times 818$
355	18·84	126,025	$\times 282$	$\times 795$
360	18·97	129,600	$\times 278$	$\times 773$
365	19·10	133,225	$\times 274$	$\times 751$
370	19·24	136,900	$\times 270$	$\times 730$
375	19·36	140,625	$10^{-5} \times 267$	$10^{-8} \times 711$
380	19·49	144,400	$\times 263$	$\times 692$
385	19·62	148,225	$\times 260$	$\times 674$
390	19·75	152,100	$\times 256$	$\times 657$
395	19·87	156,025	$\times 253$	$\times 641$
400	20·00	160,000	$10^{-5} \times 250$	$10^{-8} \times 625$
405	20·12	164,025	$\times 247$	$\times 610$
410	20·25	168,100	$\times 244$	$\times 595$
415	20·37	172,225	$\times 241$	$\times 581$
420	20·49	176,400	$\times 238$	$\times 567$

TABLE I.—(*Continued*).

Number, L.	Square-root, $\sqrt{L}$.	Square, L^2.	Reciprocal, 1/L.	Square of Reciprocal, $1/L^2$.
425	20·62	180,625	$10^{-5} \times 235$	$10^{-8} \times 554$
430	20·74	184,900	× 233	× 541
435	20·86	189,225	× 230	× 528
440	20·98	193,600	× 227	× 516
445	21·10	198,025	× 225	× 505
450	21·21	202,500	$10^{-5} \times 222$	$10^{-8} \times 494$
455	21·33	207,025	× 220	× 483
460	21·45	211,600	× 217	× 472
465	21·56	216,225	× 215	× 462
470	21·68	220,900	× 213	× 453
475	21·79	225,625	$10^{-5} \times 211$	$10^{-8} \times 443$
480	21·91	230,400	× 208	× 433
485	22·02	235,225	× 206	× 424
490	22·14	240,100	× 204	× 416
495	22·25	245,025	× 202	× 408
500	22·36	250,000	$10^{-5} \times 200$	$10^{-8} \times 400$
505	22·47	255,025	× 198	× 392
510	22·58	260,100	× 196	× 385
515	22·69	265,225	× 194	× 378
520	22·80	270,400	× 192	× 370
525	22·91	275,625	$10^{-5} \times 190$	$10^{-8} \times 363$
530	23·02	280,900	× 189	× 356
535	23·13	286,225	× 187	× 350
540	23·24	291,600	× 185	× 343
545	23·35	297,025	× 184	× 337
550	23·45	302,500	$10^{-5} \times 182$	$10^{-8} \times 331$
555	23·56	308,025	× 180	× 325
560	23·66	313,600	× 179	× 319
565	23·77	319,225	× 177	× 313
570	23·87	324,900	× 175	× 308
575	23·98	330,625	$10^{-5} \times 174$	$10^{-8} \times 302$
580	24·08	336,400	× 172	× 297
585	24·19	342,225	× 171	× 292
590	24·29	348,100	× 169	× 287
595	24·39	354,025	× 168	× 282
600	24·49	360,000	$10^{-5} \times 167$	$10^{-8} \times 278$
605	24·60	366,025	× 165	× 273
610	24·70	372,100	× 164	× 269
615	24·80	378,225	× 163	× 264
620	24·90	384,400	× 161	× 260
625	25·00	390,625	$10^{-5} \times 160$	$10^{-8} \times 256$
630	25·10	396,900	× 159	× 252
635	25·20	403,225	× 158	× 248
640	25·30	409,600	× 156	× 244
645	25·40	416,025	× 155	× 240

TABLE I.—(*Continued*).

Number, L.	Square-root, $\sqrt{L}$.	Square, L^2.	Reciprocal, 1/L.	Square of Reciprocal, $1/L^2$.
650	25·50	422,500	$10^{-5} \times 154$	$10^{-8} \times 237$
655	25·59	429,025	$\times 153$	$\times 233$
660	25·69	435,600	$\times 152$	$\times 230$
665	25·79	442,225	$\times 150$	$\times 226$
670	25·88	448,900	$\times 149$	$\times 223$
675	25·98	455,625	$10^{-5} \times 148$	$10^{-8} \times 220$
680	26·08	462,400	$\times 147$	$\times 216$
685	26·17	469,225	$\times 146$	$\times 213$
690	26·27	476,100	$\times 145$	$\times 210$
695	26·36	483,025	$\times 144$	$\times 207$
700	26·46	490,000	$10^{-5} \times 143$	$10^{-8} \times 204$
705	26·55	497,025	$\times 142$	$\times 201$
710	26·65	504,100	$\times 141$	$\times 198$
715	26·74	511,225	$\times 140$	$\times 196$
720	26·83	518,400	$\times 139$	$\times 193$
725	26·93	525,625	$10^{-5} \times 138$	$10^{-8} \times 190$
730	27·02	532,900	$\times 137$	$\times 188$
735	27·11	540,225	$\times 136$	$\times 185$
740	27·20	547,600	$\times 135$	$\times 183$
745	27·29	555,025	$\times 134$	$\times 180$
750	27·39	562,500	$10^{-5} \times 133$	$10^{-8} \times 178$
755	27·48	570,025	$\times 132$	$\times 176$
760	27·57	577,600	$\times 132$	$\times 173$
765	27·66	585,225	$\times 131$	$\times 171$
770	27·75	592,900	$\times 130$	$\times 169$
775	27·84	600,625	$10^{-5} \times 129$	$10^{-8} \times 166$
780	27·93	608,400	$\times 128$	$\times 164$
785	28·02	616,225	$\times 127$	$\times 162$
790	28·11	624,100	$\times 127$	$\times 160$
795	28·20	632,025	$\times 126$	$\times 158$
800	28·28	640,000	$10^{-5} \times 125$	$10^{-8} \times 156$
805	28·37	648,025	$\times 124$	$\times 154$
810	28·46	656,100	$\times 123$	$\times 152$
815	28·55	664,225	$\times 123$	$\times 150$
820	28·64	672,400	$\times 122$	$\times 149$
825	28·72	680,625	$10^{-5} \times 121$	$10^{-8} \times 147$
830	28·81	688,900	$\times 120$	$\times 145$
835	28·90	697,225	$\times 120$	$\times 143$
840	28·98	705,600	$\times 119$	$\times 142$
845	29·07	714,025	$\times 118$	$\times 140$
850	29·15	722,500	$10^{-5} \times 118$	$10^{-8} \times 138$
855	29·24	731,025	$\times 117$	$\times 137$
860	29·33	739,600	$\times 116$	$\times 135$
865	29·41	748,225	$\times 116$	$\times 134$
870	29·50	756,900	$\times 115$	$\times 132$

TABLE I.—(*Continued*).

Number, L.	Square-root, $\sqrt{L}$.	Square, L^2.	Reciprocal, 1/L.	Square of Reciprocal, $1/L^2$.
875	29·58	765,625	$10^{-5}\times 114$	$10^{-8}\times 131$
880	29·66	774,400	$\times 114$	$\times 129$
885	29·75	783,225	$\times 113$	$\times 128$
890	29·83	792,100	$\times 112$	$\times 126$
895	29·92	801,025	$\times 112$	$\times 125$
900	30·00	810,000	$10^{-5}\times 111$	$10^{-8}\times 124$
905	30·08	819,025	$\times 110$	$\times 122$
910	30·17	828,100	$\times 110$	$\times 121$
915	30·25	837,225	$\times 109$	$\times 120$
920	30·33	846,400	$\times 109$	$\times 118$
925	30·41	855,625	$10^{-5}\times 108$	$10^{-8}\times 117$
930	30·50	864,900	$\times 108$	$\times 116$
935	30·58	874,225	$\times 107$	$\times 114$
940	30·66	883,600	$\times 106$	$\times 113$
945	30·74	893,025	$\times 106$	$\times 112$
950	30·82	902,500	$10^{-5}\times 105$	$10^{-8}\times 111$
955	30·90	912,025	$\times 105$	$\times 110$
960	30·98	921,600	$\times 104$	$\times 108$
965	31·06	931,225	$\times 104$	$\times 107$
970	31·14	940,900	$\times 103$	$\times 106$
975	31·22	950,625	$10^{-5}\times 103$	$10^{-8}\times 105$
980	31·30	960,400	$\times 102$	$\times 104$
985	31·38	970,225	$\times 102$	$\times 103$
990	31·46	980,100	$\times 101$	$\times 102$
995	31·54	990,025	$\times 100$	$\times 101$
1,000	31·62	1,000,000	$10^{-5}\times 100$	$10^{-8}\times 100$

TABLE II.—CONVERSION TABLE: SECONDS AND RADIANS.

Seconds.	Equivalent in Radians.		Square of Angle in Radians.	
	v.	log v.	v^2.	log v^2.
1	$10^{-6} \times 4{\cdot}848$	·6856	$10^{-11} \times 2{\cdot}350$	·3712
2	× 9·696	·9866	× 9·401	·9732
3	$10^{-5} \times 1{\cdot}454$	·1627	$10^{-10} \times 2{\cdot}115$	·3254
4	× 1·939	·2877	× 3·761	·5754
5	× 2·424	·3846	× 5·878	·7692
6	× 2·909	·4638	× 8·465	·9276
7	× 3·394	·5307	$10^{-9} \times 1{\cdot}152$	·0614
8	× 3·879	·5887	× 1·504	·1774
9	× 4·363	·6398	× 1·904	·2796
10	× 4·848	·6856	× 2·350	·3712
11	× 5·333	·7270	× 2·844	·4540
12	× 5·819	·7648	× 3·386	·5296
13	× 6·302	·7995	× 3·972	·5990
14	× 6·787	·8317	× 4·609	·6634
15	× 7·273	·8617	× 5·289	·7234
16	× 7·757	·8897	× 6·017	·7794
17	× 8·241	·9160	× 6·792	·8320
18	× 8·728	·9409	× 7·617	·8818
19	× 9·212	·9644	× 8·487	·9288
20	× 9·696	·9866	× 9·401	·9732
21	$10^{-4} \times 1{\cdot}018$	·0078	$10^{-8} \times 1{\cdot}036$	·0156
22	× 1·067	·0280	× 1·138	·0560
23	× 1·115	·0473	× 1·244	·0946
24	× 1·163	·0658	× 1·354	·1316
25	× 1·212	·0835	× 1·469	·1670
26	× 1·261	·1006	× 1·590	·2012
27	× 1·309	·1170	× 1·714	·2340
28	× 1·358	·1328	× 1·844	·2656
29	× 1·406	·1480	× 1·977	·2960
30	× 1·454	·1627	× 2·115	·3254
31	× 1·503	·1770	× 2·259	·3540
32	× 1·553	·1907	× 2·406	·3814
33	× 1·600	·2041	× 2·560	·4082
34	× 1·648	·2171	× 2·717	·4342
35	× 1·697	·2297	× 2·880	·4594
36	× 1·746	·2419	× 3·047	·4838
37	× 1·794	·2538	× 3·218	·5076
38	× 1·843	·2654	× 3·394	·5308
39	× 1·891	·2767	× 3·576	·5534
40	× 1·939	·2877	× 3·761	·5754
41	× 1·988	·2984	× 3·952	·5968
42	× 2·036	·3088	× 4·146	·6176

TABLE II.—(*Continued*).

Seconds.	Equivalent in Radians.		Square of Angle in Radians.	
	v.	log v.	v^2.	log v^2.
43	10^{-4} × 2·084	·3191	10^{-8} × 4·347	·6382
44	× 2·135	·3291	× 4·550	·6582
45	× 2·182	·3388	× 4·760	·6776
46	× 2·230	·3484	× 4·975	·6968
47	× 2·279	·3577	× 5·193	·7154
48	× 2·327	·3668	× 5·416	·7336
49	× 2·374	·3758	× 5·644	·7516
50	× 2·424	·3846	× 5·878	·7692
51	× 2·473	·3932	× 6·115	·7864
52	× 2·522	·4016	× 6·431	·8032
53	× 2·569	·4099	× 6·604	·8198
54	× 2·618	·4180	× 6·855	·8360
55	× 2·667	·4260	× 7·112	·8520
56	× 2·721	·4338	× 7·372	·8676
57	× 2·764	·4415	× 7·638	·8830
58	× 2·812	·4490	× 7·907	·8980
59	× 2·861	·4565	× 8·185	·9130
60	× 2·909	·4638	× 8·465	·9276
61	× 2·957	·4709	× 8·746	·9418
62	× 3·006	·4780	× 9·036	·9560
63	× 3·054	·4849	× 9·328	·9698
64	× 3·103	·4918	× 9·629	·9836
65	× 3·152	·4985	× 9·931	·9970
66	× 3·200	·5051	10^{-7} × 1·023	·0102
67	× 3·248	·5117	× 1·055	·0234
68	× 3·297	·5181	× 1·086	·0362
69	× 3·345	·5244	× 1·119	·0488
70	× 3·394	·5307	× 1·152	·0614
71	× 3·443	·5369	× 1·185	·0738
72	× 3·490	·5429	× 1·218	·0858
73	× 3·539	·5489	× 1·252	·0978
74	× 3·588	·5548	× 1·287	·1096
75	× 3·637	·5607	× 1·322	·1214
76	× 3·684	·5664	× 1·358	·1328
77	× 3·734	·5721	× 1·394	·1442
78	× 3·782	·5777	× 1·430	·1554
79	× 3·830	·5832	× 1·467	·1664
80	× 3·879	·5887	× 1·504	·1774
81	× 3·927	·5941	× 1·543	·1882
82	× 3·975	·5994	× 1·584	·1998
83	× 4·022	·6047	× 1·619	·2094
84	× 4·072	·6099	× 1·659	·2198
85	× 4·121	·6150	× 1·698	·2300

TABLE II.—(*Continued*).

Seconds.	Equivalent in Radians.		Square of Angle in Radians.	
	v.	log v.	v^2.	log v^2.
86	$10^{-4} \times 4{\cdot}170$	·6201	$10^{-7} \times 1{\cdot}739$	·2402
87	$\times 4{\cdot}218$	·6251	$\times 1{\cdot}779$	·2502
88	$\times 4{\cdot}267$	·6301	$\times 1{\cdot}821$	·2602
89	$\times 4{\cdot}315$	·6350	$\times 1{\cdot}862$	·2700
90	$\times 4{\cdot}363$	·6398	$\times 1{\cdot}904$	·2796
91	$\times 4{\cdot}412$	·6446	$\times 1{\cdot}946$	·2892
92	$\times 4{\cdot}461$	·6494	$\times 1{\cdot}990$	·2988
93	$\times 4{\cdot}509$	·6541	$\times 2{\cdot}033$	·3082
94	$\times 4{\cdot}557$	·6587	$\times 2{\cdot}077$	·3174
95	$\times 4{\cdot}606$	·6633	$\times 2{\cdot}121$	·3266
96	$\times 4{\cdot}655$	·6679	$\times 2{\cdot}167$	·3358
97	$\times 4{\cdot}703$	·6724	$\times 2{\cdot}212$	·3448
98	$\times 4{\cdot}751$	·6768	$\times 2{\cdot}257$	·3536
99	$\times 4{\cdot}799$	·6812	$\times 2{\cdot}303$	·3624
100	$\times 4{\cdot}848$	·6856	$\times 2{\cdot}350$	·3712

TABLE III.—Odds in Favour of an Error being less than x Times the Average Error.

(*The values are only approximate, but are sufficiently exact for practical purposes.*)

x.	Odds in Favour.	x	Odds in Favour.
0·1	0·070 : 1	2·1	9·6 : 1
0·2	0·12 : 1	2·2	11·5 : 1
0·3	0·23 : 1	2·3	13·8 : 1
0·4	0·33 : 1	2·4	16·8 : 1
0·5	0·45 : 1	2·5	20·4 : 1
0·6	0·58 : 1	2·6	25·0 : 1
0·7	0·74 : 1	2·7	30·9 : 1
0·8	0·90 : 1	2·8	37·4 : 1
0·9	1·05 : 1	2·9	46·4 : 1
1·0	1·35 : 1	3·0	54·0 : 1
1·1	1·63 : 1	3·1	78·3 : 1
1·2	1·96 : 1	3·2	95·2 : 1
1·3	2·31 : 1	3·3	117 : 1
1·4	2·76 : 1	3·4	142 : 1
1·5	3·31 : 1	3·5	174 : 1
1·6	3·95 : 1	3·6	237 : 1
1·7	4·72 : 1	3·7	332 : 1
1·8	5·53 : 1	3·8	416 : 1
1·9	6·65 : 1	3·9	525 : 1
2·0	8·02 : 1	4·0	666 : 1

INDEX.

References are to pages.

A.

B.

F.

G.

H.

I.

J.

L.

M.

U.

V.

W.

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